Stable ordered-union ultrafilters

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Outline

Selective ultrafilters

- Ramsey's theorem for pairs states that if $c : [\omega]^2 \to 2$ is any coloring, then there exists $H \in [\omega]^{\aleph_0}$ such that c is constant on $[H]^2$.

Definition

$[X]^2$ denotes the collection of all unordered pairs from X – that is $[X]^2 = \{\{x, y\} \subseteq X : x \neq y\}$.

Definition

*An ultrafilter $\mathcal{U}$ on ω is said to be **selective** if for every $c : [\omega]^2 \to 2$, there exists $H \in \mathcal{U}$ such that c is constant on $[H]^2$.*

Definition

*A function $f : \omega \to \omega$ is **canonical on** a subset $A \subseteq \omega$ if f is either constant or one-to-one on A.*

Definition

*An ultrafilter $\mathcal{U}$ on ω is called a **P-point** if for every function $g : \omega \to \omega$, there exists $A \in \mathcal{U}$ such that g is either constant or finite-to-one on A.*
*An ultrafilter $\mathcal{U}$ on ω is called a **Q-point** if for every finite-to-one function $f : \omega \to \omega$, there exists $A \in \mathcal{U}$ such that f is one-to-one on A.*

Theorem

The following are equivalent for any ultrafilter $\mathcal{U}$ on ω:

1. *$\mathcal{U}$ is selective;*
2. *for each $1 \leq n, k < \omega$ and $c : [\omega]^n \to k$, there is an $A \in \mathcal{U}$ such that c is constant on $[A]^n$;*
3. *for every function $f : \omega \to \omega$, there exists $A \in \mathcal{U}$ such that f is canonical on A;*
4. *whenever $\mathcal{X} \subseteq [\omega]^{\aleph_0}$ is analytic, there exists $A \in \mathcal{U}$ such that either $[A]^{\aleph_0} \subseteq \mathcal{X}$ or $[A]^{\aleph_0} \cap \mathcal{X} = \emptyset$;*
5. *$\mathcal{U}$ is both a P-point and a Q-point.*

- The partition property in item (4) can be further strengthened in the presence of large cardinals to include all subsets of $[\omega]^{\aleph_0}$ in $\mathbf{L}(\mathbb{R})$.
- $([\omega]^{\omega}, \subseteq^{*})$ is a countably closed forcing, and hence it does not add any reals. If $\mathcal{U} \subseteq [\omega]^{\omega}$ is a generic filter for this forcing over some model $\mathbf{V}$, then $\mathcal{U}$ is a selective ultrafilter in $\mathbf{V}[G]$.

Theorem (Todorcevic)

Assume that there is a supercompact cardinal. $\mathcal{U}$ is a selective ultrafilter on ω if and only if $\mathcal{U}$ is a generic filter for the forcing $([\omega]^{\omega}, \subseteq^{*})$ over $\mathbf{L}(\mathbb{R})$.

Definition

*Let $\mathcal{U}$ be an ultrafilter on ω. The **selectivity game on $\mathcal{U}$**, denoted $\Game^{\mathtt{Sel}}(\mathcal{U})$, is a two player perfect information game in which Players I and II alternatively choose A_i and n_i respectively, where $A_i \in \mathcal{U}$ and $n_i \in A_i$. Together they construct the sequence*

$$A_0, n_0, A_1, n_1, \dots,$$

where each $A_i \in \mathcal{U}$ has been played by Player I and $n_i \in A_i$ has been chosen by Player II in response. Player II wins if and only if $\{n_i : i < \omega\} \in \mathcal{U}$.

Theorem (Galvin; McKenzie)

An ultrafilter $\mathcal{U}$ on ω is selective if and only if Player I does not have a winning strategy in $\Game^{\mathtt{Sel}}(\mathcal{U})$.

Existence and non-existence

Theorem (Kunen)

There are no selective ultrafilters when $\aleph_2$ random reals are added to any model of $\mathrm{ZFC} + \mathrm{CH}$.

Theorem (Shelah)

It is consistent that there are no P-points.

Theorem (Chodounsky and Guzman)

There are no P-points in Silver models.

Theorem (Miller)

There are no Q-points in the Laver model nor in the Miller model.

Theorem (Ketonen)

If $\mathfrak{d} = \mathfrak{c}$, then there are P-points.

Theorem (Ketonen)

If $\mathfrak{d} = \aleph_1$, then there are Q-points.

Theorem (Canjar)

If $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$, then there are selective ultrafilters.

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Theorem (Canjar)

If $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$, then there are selective ultrafilters.

Question

Is it consistent that there are no P-points and no Q-points?

- If there are no P-points and no Q-points, then $\aleph_1 < \mathfrak{d} < \mathfrak{c}$. In particular, $\mathfrak{c} \geq \aleph_3$.

Hindman's Theorem

Theorem (Hindman's Theorem)

Whenever ω is partitioned into finitely many pieces, then one of the pieces contains all distinct sums from some infinite set.

- Hindman's theorem works in many semigroups. My focus here will be on $(\mathrm{FIN}, \cup)$.

Definition

FIN *denotes the collection of non-empty finite subsets of* ω*. For* $s, t \in \mathrm{FIN}$*, write* $s <_{\mathfrak{b}} t$ *to mean* $\max(s) < \min(t)$*.* $X \subseteq \mathrm{FIN}$ *is called a* ***block sequence*** *if* X *is non-empty and it is linearly ordered by the relation* $<_{\mathfrak{b}}$*.*

Definition

For $1 \leq \alpha \leq \omega$ and $A \subseteq \mathrm{FIN}$, $A^{[\alpha]}$ denotes the collection of all block sequences of length α from A and $A^{[<\alpha]}$ is the collection of all block sequences of length $< \alpha$ from A

Definition

If X is a block sequence, then $[X]$ denotes the collection of finite non-empty unions from X. In other words,

$$[X] = \left\{ \bigcup Y : Y \in X^{[<\omega]} \right\}.$$

Theorem (Hindman's Theorem)

For every $c : \mathrm{FIN} \to \{0, 1\}$, there exists $X \in \mathrm{FIN}^{[\omega]}$ such that c is constant on $[X]$.

- This version of Hindman's theorem implies the version for $(\omega, +)$ via the map $s \mapsto \sum_{n \in s} 2^n$, for $s \in \mathrm{FIN}$.
- There are four types of ultrafilters one can associate with Hindman's theorem. We will start with the weakest.

Definition

Define $\mathcal{I}_{\mathtt{Hindman}} = \left\{A \subseteq \mathrm{FIN} : \neg \exists X \in \mathrm{FIN}^{[\omega]} \, [[X] \subseteq A]\right\}$. *Hindman's theorem implies that* $\mathcal{I}_{\mathtt{Hindman}}$ *is a (proper, non-principal) ideal on* FIN.

- For an ultrafilter $\mathcal{H}$ on FIN, $\mathcal{H} \cap \mathcal{I}_{\mathtt{Hindman}} = \emptyset$ if and only if for every $A \in \mathcal{H}$, there exists $X \in \mathrm{FIN}^{[\omega]}$ with $[X] \subseteq A$.
- Such $\mathcal{H}$ exist by Zorn's Lemma.

- Next, Hindman's theorem is closely related to idempotents in $\beta\mathrm{FIN}$.

Definition

Let $\gamma\mathrm{FIN} = \{\mathcal{H} \in \beta\mathrm{FIN} : \forall k \in \omega\, [\{s \in \mathrm{FIN} : k < \min(s)\} \in \mathcal{H}]\}$.
For $\mathcal{G}$ *and* $\mathcal{H}$ *in* $\gamma\mathrm{FIN}$ *define*

$$\mathcal{G} \cup \mathcal{H} = \{A \subseteq \mathrm{FIN} : \{s \in \mathrm{FIN} : \{t \in \mathrm{FIN} : s <_{\mathrm{b}} t \text{ and } s \cup t \in A\} \in \mathcal{H}\} \in \mathcal{G}\}.$$

- $(\gamma\mathrm{FIN}, \cup)$ is a compact semigroup.
- By a theorem of Ellis, $(\gamma\mathrm{FIN}, \cup)$ has an **idempotent** – i.e. an $\mathcal{H}$ such that $\mathcal{H} \cup \mathcal{H} = \mathcal{H}$.
- If $\mathcal{H}$ is any idempotent in $(\gamma\mathrm{FIN}, \cup)$, then $\mathcal{H} \cap \mathcal{I}_{\mathtt{Hindman}} = \emptyset$.

Definition (Blass, 1987)

*An ultrafilter $\mathcal{H}$ on FIN is called **ordered-union** if for every $c : \mathrm{FIN} \to \{0, 1\}$, there exists $X \in \mathrm{FIN}^{[\omega]}$ such that $[X] \in \mathcal{H}$ and c is constant on $[X]$*

- $\mathcal{H}$ is ordered-union if and only if for every $A \in \mathcal{H}$, there exists $X \in \mathrm{FIN}^{[\omega]}$ such that $[X] \in \mathcal{H}$ and $[X] \subseteq A$.
- Every ordered-union ultrafilter is idempotent.
- Unlike idempotents, the existence of ordered-union ultrafilters is not a theorem of ZFC.

- Ramsey's theorem and Hindman's theorem have a common generalization, called the Milliken–Taylor theorem.

Theorem (Milliken–Taylor Theorem)

For any $c : \mathrm{FIN}^{[2]} \to \{0, 1\}$, *there exists* $X \in \mathrm{FIN}^{[\omega]}$ *such that* c *is constant on* $[X]^{[2]}$.

Definition (Blass, 1987)

An ultrafilter $\mathcal{H}$ *on* FIN *is called* ***stable ordered-union*** *if for every* $c : \mathrm{FIN}^{[2]} \to \{0, 1\}$, *there exists* $X \in \mathrm{FIN}^{[\omega]}$ *such that* $[X] \in \mathcal{H}$ *and* c *is constant on* $[X]^{[2]}$.

- Hence, we have the following implications for an ultrafilter $\mathcal{H}$ on FIN:

$$\mathcal{H} \text{ is stable ordered-union } \overset{\text{(I)}}{\Longrightarrow} \mathcal{H} \text{ is ordered-union } \overset{\text{(II)}}{\Longrightarrow}$$

$$\mathcal{H} \text{ is an idempotent in } (\gamma\mathrm{FIN}, \cup) \overset{\text{(III)}}{\Longrightarrow} \mathcal{H} \cap \mathcal{I}_{\mathtt{Hindman}} = \emptyset.$$

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$$\mathcal{H} \text{ is an idempotent in } (\gamma\mathrm{FIN}, \cup) \overset{(III)}{\Longrightarrow} \mathcal{H} \cap \mathcal{I}_{\mathtt{Hindman}} = \emptyset.$$

- It is not hard to show that (II) and (III) cannot be reserved.

Question (Blass, 1980s)

Is every ordered-union ultrafilter stable?

Definition

A function $f : \mathrm{FIN} \to \omega$ *is* ***canonical on*** *a subset* $A \subseteq \mathrm{FIN}$ *if one of the following statements hold:*

1. $\forall s, t \in A\, [f(s) = f(t)]$;
2. $\forall s, t \in A\, [f(s) = f(t) \leftrightarrow \min(s) = \min(t)]$;
3. $\forall s, t \in A\, [f(s) = f(t) \leftrightarrow \max(s) = \max(t)]$;
4. $\forall s, t \in A\, [f(s) = f(t) \leftrightarrow (\min(s) = \min(t) \wedge \max(s) = \max(t))]$;
5. $\forall s, t \in A\, [f(s) = f(t) \leftrightarrow s = t]$.

Definition

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3. $\forall s, t \in A\, [f(s) = f(t) \leftrightarrow \max(s) = \max(t)]$;
4. $\forall s, t \in A\, [f(s) = f(t) \leftrightarrow (\min(s) = \min(t) \wedge \max(s) = \max(t))]$;
5. $\forall s, t \in A\, [f(s) = f(t) \leftrightarrow s = t]$.

Definition

Let $X, Y \in \mathrm{FIN}^{[\omega]}$. Y *is said to* ***refine*** X *if* $\forall i \in \omega\, [Y(i) \in [X]]$. *We write* $Y \leq X$ *to denote this relation.* Y *is said to* ***almost refine*** X *if* $\forall^{\infty} i \in \omega\, [Y(i) \in [X]]$. *This relation is denoted by* $Y \leq^{*} X$.

Theorem (Blass, 1987)

The following are equivalent for any ultrafilter $\mathcal{H}$ *on* FIN*:*

1. $\mathcal{H}$ *is stable ordered-union;*
2. *for each* $1 \leq n, k < \omega$ *and* $c : \mathrm{FIN}^{[n]} \rightarrow k$*, there is an* $X \in \mathrm{FIN}^{[\omega]}$ *such that* $[X] \in \mathcal{H}$ *and* c *is constant on* $[X]^{[n]}$*;*
3. *for every function* $f : \mathrm{FIN} \rightarrow \omega$*, there exists* $X \in \mathrm{FIN}^{[\omega]}$ *such that* $[X] \in \mathcal{H}$ *and* f *is canonical on* $[X]$*;*
4. *whenever* $\mathcal{X} \subseteq \mathrm{FIN}^{[\omega]}$ *is analytic, there exists* $X \in \mathrm{FIN}^{[\omega]}$ *such that* $[X] \in \mathcal{H}$ *and either* $[X]^{[\omega]} \subseteq \mathcal{X}$ *or* $[X]^{[\omega]} \cap \mathcal{X} = \emptyset$*;*
5. $\mathcal{H}$ *is ordered-union and for every sequence* $\langle X_n : n \in \omega \rangle$ *with the property that for all* $n \in \omega$*,* $X_n \in \mathrm{FIN}^{[\omega]}$ *and* $[X_n] \in \mathcal{H}$*, there exists* $Y \in \mathrm{FIN}^{[\omega]}$ *such that* $\forall n \in \omega\, [Y \leq^* X_n]$ *and* $[Y] \in \mathcal{H}$*.*

- The partition property given by item (4) can be strengthened further in the presence of large cardinals to cover all subsets of $\mathrm{FIN}^{[\omega]}$ in $\mathbf{L}(\mathbb{R})$.
- $(\mathrm{FIN}^{[\omega]}, \leq^*)$ is a countably closed forcing notion, and hence, it does not add any new reals.
- If $G \subseteq \mathrm{FIN}^{[\omega]}$ is a generic filter over some transitive universe $\mathbf{V}$, then it is easy to see that $\mathcal{H} = \{A \subseteq \mathrm{FIN} : \exists X \in G\,[[X] \subseteq A]\}$ is a stable ordered-union ultrafilter in $\mathbf{V}[G]$. We will say that $\mathcal{H}$ is the **ultrafilter added by** G if it has this form.

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Theorem (Todorcevic)

Assume that there is a supercompact cardinal. $\mathcal{H}$ is a stable ordered-union ultrafilter on FIN if and only if $\mathcal{H}$ is added by some generic filter for the forcing $(\mathrm{FIN}^{[\omega]}, \leq^*)$ over $\mathbf{L}(\mathbb{R})$.

Definition

*Let $\mathcal{H}$ be any ultrafilter on FIN. The **stability game on** $\mathcal{H}$, denoted $\Game^{\mathtt{Stab}}(\mathcal{H})$, is a two player game in which Players I and II alternatively choose sets A_i and s_i respectively, where $A_i \in \mathcal{H}$ and $s_i \in A_i$. During a run of the game, they construct the sequence*

$$A_0, s_0, A_1, s_1, \dots,$$

where each $A_i \in \mathcal{H}$ has been played by Player I and $s_i \in A_i$ has been chosen by Player II in response. Player II wins this run if and only if $\forall i < j < \omega \left[s_i <_{\mathrm{b}} s_j \right]$ and $[\{s_i : i < \omega\}] \in \mathcal{H}$.

Theorem (see Lemma 2.13 of [2])

An ultrafilter $\mathcal{H}$ on FIN is stable ordered-union if and only if Player I does not have a winning strategy in $\Game^{\mathtt{Stab}}(\mathcal{H})$.

Rudin-Keisler ordering on ultrafilters

Definition

Let $\mathcal{F}$ be a filter on X and $\mathcal{G}$ a filter on Y. $\mathcal{F}$ is said to be ***Rudin-Keisler below*** *$\mathcal{G}$, written $\mathcal{F} \leq_{RK} \mathcal{G}$, if there exists a function $f : Y \to X$ such that for every $A \subseteq X$,*

$$A \in \mathcal{F} \iff f^{-1}(A) \in \mathcal{G}$$

$\mathcal{F}$ and $\mathcal{G}$ are ***Rudin-Keisler equivalent****, written $\mathcal{F} \equiv_{RK} \mathcal{G}$ if $\mathcal{F} \leq_{RK} \mathcal{G}$ and $\mathcal{G} \leq_{RK} \mathcal{F}$.*

Fact

If $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on ω, then $\mathcal{U} \equiv_{RK} \mathcal{V}$ if and only if there is a permutation $e : \omega \to \omega$ such that $\mathcal{U} = \{e''B : B \in \mathcal{V}\}$.

- If $\mathcal{U} \equiv_{RK} \mathcal{V}$, we say $\mathcal{U}$ and $\mathcal{V}$ are RK-isomorphic.

Fact

If $\mathcal{U}$ and $\mathcal{V}$ are selective and $\mathcal{U} \leq_{RK} \mathcal{V}$, then $\mathcal{U} \equiv_{RK} \mathcal{V}$. Therefore, selective ultrafilters are RK-minimal. Conversely, every RK-minimal ultrafilter on ω is selective.

Definition

Let $\mathcal{H}$ *be an ultrafilter on* FIN. *Define*

$$\mathcal{H}_{\min} = \{M \subseteq \omega : \{s \in \mathrm{FIN} : \min(s) \in M\} \in \mathcal{H}\}$$
$$\mathcal{H}_{\max} = \{M \subseteq \omega : \{s \in \mathrm{FIN} : \max(s) \in M\} \in \mathcal{H}\}.$$

- $\mathcal{H}_{\min}$ and $\mathcal{H}_{\max}$ are ultrafilters on ω, and the maps $\min : \mathrm{FIN} \to \omega$ and $\max : \mathrm{FIN} \to \omega$ witness that $\mathcal{H}_{\min}, \mathcal{H}_{\max} \leq_{RK} \mathcal{H}$.

Theorem (Blass and Blass and Hindman)

Let $\mathcal{H}$ *be an ordered-union ultrafilter on* FIN. *Then* $\mathcal{H}_{\min}$ *and* $\mathcal{H}_{\max}$ *are selective ultrafilters on* ω *such that* $\mathcal{H}_{\min} \not\equiv_{RK} \mathcal{H}_{\max}$.

Corollary

It is consistent that there are no ordered-union ultrafilters.

Theorem (Eisworth)

Stable ordered-union ultrafilters exist if $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$.

A question of Blass

- The existence of a stable ordered-union ultrafilter guarantees the existence of at least two RK-non-isomorphic selective ultrafilters.

Theorem (Blass)

Assume CH, *and let* $\mathcal{U}$ *and* $\mathcal{V}$ *be selective ultrafilters such that* $\mathcal{U} \not\equiv_{RK} \mathcal{V}$. *Then there is a stable ordered-union ultrafilter* $\mathcal{H}$ *such that* $\mathcal{H}_{\max} = \mathcal{U}$ *and* $\mathcal{H}_{\min} = \mathcal{V}$.

Question (Blass, 1987)

Does the existence of at least two non-isomorphic selective ultrafilters imply the existence of a stable ordered-union ultrafilter?

Theorem (Raghavan and Steprāns [2], 2023)

There is a model of ZFC with $2^{\aleph_0}$ pairwise non-isomorphic selective ultrafilters on ω, but no stable ordered-union ultrafilters on FIN.

Question

Is it consistent to have $2^{\aleph_0}$ pairwise non-isomorphic selective ultrafilters on ω, but no ordered-union ultrafilters on FIN?

- Blass characterized all ultrafilters that are RK below a stable ordered-union ultrafilter.

Definition

For $A \subseteq \omega \times \omega$ *and* $m \in \omega$, $A[m] = \{n \in \omega : \langle m, n \rangle \in A\}$. *Let* $\mathcal{U}$ *and* $\mathcal{V}$ *be ultrafilters on* ω. *Define*

$$\mathcal{U} \otimes \mathcal{V} = \{A \subseteq \omega \times \omega : \{m \in \omega : A[m] \in \mathcal{V}\} \in \mathcal{U}\}.$$

It is easily seen that $\mathcal{U} \otimes \mathcal{V}$ *is an ultrafilter on* $\omega \times \omega$.

Theorem (Blass [1])

Suppose that $\mathcal{H}$ is a stable ordered-union ultrafilter on FIN. If $\mathcal{U}$ is an ultrafilter on ω such that $\mathcal{U} \leq_{RK} \mathcal{H}$, then $\mathcal{U} \equiv_{RK} \mathcal{H}$, or $\mathcal{U} \equiv_{RK} \mathcal{H}_{\min} \otimes \mathcal{H}_{\max}$, or $\mathcal{U} \equiv_{RK} \mathcal{H}_{\min}$, or $\mathcal{U} \equiv_{RK} \mathcal{H}_{\max}$.

Corollary (Blass)

If $\mathcal{H}$ is a stable ordered-union ultrafilter on FIN and $\mathcal{K}$ is any ultrafilter on FIN such that $\mathcal{K} \cap \mathcal{I}_{\mathtt{Hindman}} = \emptyset$ and $\mathcal{K} \leq_{RK} \mathcal{H}$, then $\mathcal{K} \equiv_{RK} \mathcal{H}$. In particular, stable-ordered union ultrafilters are RK-minimal among all idempotents in $(\gamma\mathrm{FIN}, \cup)$.

Question

Suppose $\mathcal{H}$ is an idempotent in $(\gamma\mathrm{FIN}, \cup)$ which is RK-minimal among all idempotents in $(\gamma\mathrm{FIN}, \cup)$. Is $\mathcal{H}$ ordered-union?

Theorem (Shelah [3])

There is a model of ZFC *with a unique P-point up to RK-isomorphism.*

Question

Is it consistent that there is a unique stable ordered-union ultrafilter up to RK-isomorphism?

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