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ABSTRACT. In their work on spreading models in Banach spaces, Dilworth, Odell, and Sari [4] introduced the notion of a Suslin lower semi-lattice, a seemingly slight weakening of the notion of a Suslin tree. They posed several problems of a set theoretic nature regarding their notion. In this paper, we make a systematic study of the notion of Suslin lower semi-lattice, answering some of the questions raised by Dilworth, Odell, and Sari.

1. INTRODUCTION

Definition 1. $\langle L, < \rangle$ is a partial order if $\forall x, y, z \in L [(x < y \land y < z) \implies x < z]$ and $\forall x \in L [x \not< x]$. As usual, $x \leq y$ abbreviates $x < y \lor x = y$. A partial order $\langle L, < \rangle$ is called a *lower semi-lattice* if every $x, y \in L$ have a greatest lower bound $x \land y$. A lower semi-lattice is called *Suslin* if

- (1) $\langle L, \langle \rangle$ is well founded
- (2) L is uncountable
- (3) L does not contain any uncountable chains or uncountable set of pairwise incomparable elements (which we will abbreviate to *p.i.e.*).

We have used the term "p.i.e." in (3) instead of the term "antichain" in order to avoid conflict with the conventional usage of that term for forcing notions. Suslin lower semi-lattices are "lattice analogues" of Suslin trees. Any normal tree has a natural notion of greatest lower bound defined on it, and so, any Suslin tree is an example of a Suslin lower semi-lattice.

Dilworth, Odell, and Sari [4] considered this notion during their study of spreading models of Banach spaces. As this notion is so similar to that of a Suslin tree, one wants to reexamine with respect to this notion the classical questions about the existence of Suslin trees, and moreover, since this notion seems to be only a slight weakening of the notion of a Suslin tree, one wants to know whether the existence of a Suslin lower semi-lattice implies the existence of a Suslin tree. In particular, the following questions suggest themselves.

Question 2. Does ZFC imply the existence of a Suslin lower semi-lattice? (Problem 1.14 of [4])

Question 3. Does ZFC + CH imply the existence of a Suslin lower semi-lattice?

Date: May 17, 2012.

Key words and phrases. lattice, Suslin tree, P-ideal dichotomy, iterated forcing.

First author partially supported by Grants-in-Aid for Scientific Research for JSPS Fellows No. $23{\cdot}01017.$

The second author is supported by FY 2010 Researcher Exchange Program between JSPS and NSERC, Grant-in-Aid for Scientific Research (C) 22540124 Japan Society for the Promotion of Science, and Leading Young Researchers of Shizuoka University 2011.

Question 4. Does the existence of a Suslin lower semi-lattice imply the existence of a Suslin tree?

Definition 5. Let $\langle L, < \rangle$ be any well-founded partial order. This means that there is a rank function defined on $\langle L, < \rangle$. Given $x \in L$, let $\operatorname{ht}(x)$ denote the rank of x. $\operatorname{ht}(x)$ will be called the *height of* x. For an ordinal α , L_{α} denotes $\{x \in L : \operatorname{ht}(x) = \alpha\}$, and $L_{<\alpha}$ denotes $\bigcup_{\beta < \alpha} L_{\beta}$. Finally, the *height of* L, $\operatorname{ht}(L)$, is the least α such that $L_{\alpha} = 0$.

Note that L_{α} is always a p.i.e. and hence $|L_{\alpha}| \leq \omega$ for a Suslin *L*. Therefore for each $\alpha < \omega_1, L_{<\alpha}$ is countable. Since we are primarily concerned with the existence of Suslin lower semi-lattices under various conditions, we may assume that ht $(L) = \omega_1$. This is because if there a Suslin *L*, then there is one of height ω_1 . Also, we may assume that for each $x \in L$, $\{y \in L : y \geq x\}$ is uncountable. This is because for any Suslin *L*, $\{x \in L : |\{y \in L : y \geq x\}| \leq \omega\}$ is countable.

It is well known that the existence of a Suslin tree can be proved neither from ZFC nor from ZFC + CH. So if Question 4 has a positive answer, then both Questions 2 and 3 have negative answers. Dilworth, Odell, and Sari [4] purport to give an example showing that Question 3 has a positive answer in some remarks following the statement of Problem 1.14. It is easily seen, however, that this example is not well-founded.

In Sections 2 and 6 of this paper we give negative answers to Questions 2 and 4 respectively. In Section 3, we use the P-ideal dichotomy of Todorčević to give partial answers to Question 3. We show that the P-ideal dichotomy implies that if there is a Suslin lower semi-lattice, then there is one which a substructure of $\langle \mathcal{P}(\omega), \subsetneq, \cap \rangle$. This of some interest because it is impossible to have $L \subset \mathcal{P}(\omega)$ such that $\langle L, \subsetneq \rangle$ is a Suslin tree. Therefore, if the P-ideal dichotomy fails to rule out the existence of Suslin lower semi-lattices, then this must be because there are some that are fundamentally different from Suslin trees.

However, in Section 4, we show that it is consistent to have a Suslin lower semilattice that is a substructure of $\langle \mathcal{P}(\omega), \subsetneq, \cap \rangle$. In fact, such an object can always be added in a generic extension. The statement expressing the existence of such an $L \subset \mathcal{P}(\omega)$ is a Σ_2^2 sentence of a specific kind. It is shown in [5] that in the presence of large cardinals, any Σ_2^2 sentence of this specific variety which is consistent, follows from \Diamond . As large cardinals ought to be irrelevant here, one would expect that the construction of a Suslin lower semi-lattice which is a substructure of $\langle \mathcal{P}(\omega), \subsetneq, \cap \rangle$ can be carried out from \Diamond . This is done in Section 5.

In Section 6 we use the fact that it is possible to have a Suslin lower semi-lattice that is so very different from a Suslin tree to produce a model where there is a Suslin lower semi-lattice, but all Aronszajn trees are special.

We now make brief remarks on some variations on the notion of a Suslin lower semi-lattice and on the history of their study. If the requirement that least upper bounds exist is dropped, then there is a ZFC example, namely Sierpinski's well known partial order. Let $\mathbb{P} = \{x_{\alpha} : \alpha < \omega_1\}$ be a collection of distinct real numbers. Define \triangleleft on \mathbb{P} by stipulating that $x_{\alpha} \triangleleft x_{\beta}$ iff $\alpha < \beta$ and $x_{\alpha} < x_{\beta}$. Then $\langle \mathbb{P}, \triangleleft \rangle$ is an uncountable well-founded partial order with no uncountable p.i.e. or chain. However, not all pairs of elements will have least upper bounds. If the requirement that the semi-lattice be well-founded is dropped, then several examples from ZFC + CH (and even weaker hypotheses) are known. Van Douwen and Kunen [11] produced from ZFC + CH an uncountable lower semi-lattice that is a sub-structure of $\langle \mathcal{P}(\omega), \subsetneq, \cap \rangle$ with no uncountable p.i.e. or chain. Todorčević [9] showed how to get such an example from $\mathfrak{b} = \aleph_1$. These examples are not well-founded. Baumgartner and Komjáth [2] produced an uncountable Boolean sub-algebra of $\mathcal{P}(\omega)$ with no uncountable p.i.e. or chain from \Diamond and Shelah [7] weakened the hypothesis to CH. Moreover, Baumgartner [1] produced a model where every uncountable suborder of $\langle \mathcal{P}(\omega), \subsetneq \rangle$ either contains an uncountable p.i.e. or chain. Finally, we note that by results of Harrington, Marker, and Shelah [6], any Borel partial order is either the union of countably many chains or else contains a perfect set of pairwise incomparable elements; so there are no "nicely definable" examples of any of the above mentioned phenomena.

The only question that remains open is Question 3. We think there is enough evidence to make the following

Conjecture 6. ZFC + CH implies the existence of a Suslin lower semi-lattice.

This paper is composed of the work of two authors as follows. The results in Sections 2, 3, 4 and 6 are due to the first author and were obtained in late 2008/early 2009. The first author benefited from several conversations with Steprāns and Todorčević. Section 5 is the work of the second author and was carried out in the summer of 2010.

2. MA_{\aleph_1} and Suslin Lower Semi-lattices

In this section we answer Question 2 by showing that there are no Suslin lower semi-lattices under MA_{\aleph_1} . This is a more or less trivial corollary of a well-known theorem of Todorčević.

The following lemma will play a crucial role in our analysis of Suslin lower semilattices throughout the paper. For a set $A \subset L$, $x \in L$ is called an *upper bound for* A if $\forall a \in A [a \leq x]$.

Lemma 7. Let $\langle L, < \rangle$ be any well-founded lower semi-lattice. Let $A \subset L$ be a subset with an upper bound. Then A has a least upper bound.

Proof. Let x be an upper bound for A of minimal rank. Let y be any other upper bound of A. Note that $x \wedge y$ is also an upper bound for A. Therefore, $x \wedge y = x$, whence $x \leq y$.

This simple observation together with a theorem of Todorčević [8] can be used to show that there are no Suslin lower semi-lattices under $MA_{\aleph 1}$.

Definition 8. Let $\langle L, < \rangle$ be a well-founded lower semi-lattice. For a set $A \subset L$ with an upper bound, let $\bigvee A$ denote its least upper bound.

Theorem 9. MA_{\aleph_1} implies that there are no Suslin lower semi-lattices.

Proof. By a theorem of Todorčević [8], under $\operatorname{MA}_{\aleph_1}$ every uncountable partial order $\langle L, < \rangle$ either contains an uncountable p.i.e. or else an uncountable set each of whose countable subsets has an upper bound in L. Applying this theorem to a Suslin L, we get the second alternative. Namely, an uncountable subset $X \subset L$ such that every countable subset of X has an upper bound in L. For each $\alpha < \omega_1$, put $x_{\alpha} = \bigvee (X \cap L_{<\alpha})$. Clearly, $\{x_{\alpha} : \alpha < \omega_1\}$ is a chain. Moreover, it is uncountable because X is uncountable.

3. P-IDEAL DICHOTOMY AND SUSLIN LOWER SEMI-LATTICES

The P-ideal dichotomy (PID) is a strong combinatorial principle introduced by Todorčević (see [10]). It is a consequence of the Proper Forcing Axiom (PFA), but it is consistent with CH. On the other hand, it is still strong enough to imply many of the things that PFA does. For example, PID implies that there are no Suslin trees. So PID gives us an axiomatic method to show that certain consequences of PFA are consistent with CH, allowing us to bypass iterated forcing arguments. Hence it is natural to investigate whether PID implies that there are no Suslin lower semi-lattices as well.

Definition 10. Let X be an uncountable set. An ideal $\mathcal{I} \subset [X]^{\leq \omega}$ is called a *P-ideal* if for every countable collection $\{x_n : n \in \omega\} \subset \mathcal{I}$, there is $x \in \mathcal{I}$ such that $\forall n \in \omega \ [x_n \subset^* x]$.

All ideals are assumed to be non-principal, meaning that $[X]^{\leq \omega} \subset \mathcal{I}$. Recall the P-ideal dichotomy of Todorčević [10].

Definition 11. The P-ideal dichotomy (PID) is the following statement: For any P-ideal \mathcal{I} on an uncountable set X either

(1) There is an uncountable set $Y \subset X$ such that $[Y]^{\leq \omega} \subset \mathcal{I}$

(2) There exist $\{X_n : n \in \omega\}$ such that the X_n are pairwise disjoint, $X = \bigcup_{n \in \omega} X_n$, and $\forall n \in \omega [[X_n]^{\omega} \cap \mathcal{I} = 0].$

Theorem 12. Assume PID. Let $\langle L, < \rangle$ be a well-founded lower semi-lattice with $ht(L) = \omega_1$. Assume that for any $A \in [L]^{\leq \omega}$,

(*)
$$\left|\left\{\bigvee B: B \subset A \land B \text{ has an upper bound in } L\right\}\right| < \mathfrak{p}$$

Then $\langle L, \langle \rangle$ either has an uncountable p.i.e. or an uncountable chain.

Proof. Assume that L has no uncountable p.i.e. For each $A \in [L]^{\leq \omega}$, put $\operatorname{sups}(A) = \{ \bigvee B : B \subset A \land B$ has an upper bound in L}. Define \mathcal{I} to be

$$\left\{A \in [L]^{\leq \omega} : \forall x \in L \left[|\operatorname{pred}(x) \cap A| < \omega\right]\right\}.$$

Here $\operatorname{pred}(x)$ denotes $\{y \in L : y \leq x\}$. It is clear that \mathcal{I} is an ideal. Next, note that for any $A \in [L]^{\leq \omega}$, $A \in \mathcal{I}$ iff $\forall x \in \operatorname{sups}(A)$ [$|\operatorname{pred}(x) \cap A| < \omega$]. Indeed, if $A \notin \mathcal{I}$, then for some $x \in L$, $|\operatorname{pred}(x) \cap A| = \omega$. Put $B = \operatorname{pred}(x) \cap A$ and $y = \bigvee B \in \operatorname{sups}(A)$. Then $|A \cap \operatorname{pred}(y)| = \omega$. Now, to check that \mathcal{I} is a P-ideal, fix $\{A_n : n \in \omega\} \subset \mathcal{I}$. Without loss of generality, the A_n are pairwise disjoint. Put $A = \bigcup_{n \in \omega} A_n$. By assumption, for each $a \in \operatorname{sups}(A)$ and $n \in \omega$, $\operatorname{pred}(a) \cap A_n$ is finite. Since $|\operatorname{sups}(A)| < \mathfrak{p}$, we can find $H(n) \in [A_n]^{<\omega}$ such that for any $a \in \operatorname{sups}(A), \forall^{\infty} n \in \omega [\operatorname{pred}(a) \cap A_n \subset H(n)]$. Now, $B = \bigcup_{n \in \omega} (A_n \setminus H(n))$ is in \mathcal{I} and $\forall n \in \omega [A_n \subset^* B]$.

First, suppose that alternative (1) of PID occurs. So fix $Y \in [L]^{\omega_1}$ such that for any $x \in L$, $\operatorname{pred}(x) \cap Y$ is finite. Then $\langle Y, \langle \rangle$ is an uncountable, well-founded partial order. For any $y \in Y$, let $\operatorname{ht}_Y(y)$ denote the rank of y in $\langle Y, \langle \rangle$. If there is $y \in Y$ with $\operatorname{ht}_Y(y) \geq \omega$, then $\operatorname{pred}(y) \cap Y$ is infinite. Therefore, for any $y \in Y$, $\operatorname{ht}_Y(y) < \omega$, and so there must be $n \in \omega$ such that $\{y \in Y : \operatorname{ht}_Y(y) = n\}$ is uncountable, whence L contains an uncountable p.i.e.

So alternative (1) of PID fails, and so there exist $\{X_n : n \in \omega\}$ such that the X_n are pairwise disjoint, $L = \bigcup_{n \in \omega} X_n$, and for each $n \in \omega$, $[X_n]^{\omega} \cap \mathcal{I} = 0$. Say that a set $A \subset L$ is finitely bounded if there exist $x_0, \ldots, x_i \in L$ such that $\forall a \in A \exists 0 \leq j \leq i \ [a \leq x_j]$. Fix $n \in \omega$ and $A \in [X_n]^{\omega}$. We claim that A must be finitely bounded. Suppose not. For any $x \in L$, let $\operatorname{pred}_A(x)$ denote $A \cap \operatorname{pred}(x)$. By the hypothesis that A is not finitely bounded, for any $x_0, \ldots, x_i \in \operatorname{sups}(A)$, $A \setminus (\operatorname{pred}_A(x_0) \cup \cdots \cup \operatorname{pred}_A(x_i))$ is infinite. Since $|\operatorname{sups}(A)| < \mathfrak{p}$, there is $B \in [A]^{\omega}$ such that for any $x \in \operatorname{sups}(A)$, $B \cap \operatorname{pred}_A(x)$ is finite. However, since $B \notin \mathcal{I}$, there is $x \in \operatorname{sups}(B) \subset \operatorname{sups}(A)$ such that $\operatorname{pred}(x) \cap B$ is infinite. But then, $\operatorname{pred}_A(x) \cap B$ is infinite, a contradiction.

Now, define a relation R on $L^{<\omega} \setminus \{0\}$ as follows. For $\sigma, \tau \in L^{<\omega} \setminus \{0\}, \tau R \sigma$ iff there are $\eta \in L^{<\omega} \setminus \{0\}$ and $i < |\sigma|$ such that $\forall j < |\eta| [\eta(j) < \sigma(i)]$ and $\tau =$ $(\sigma \upharpoonright i)^{\frown}(\eta)^{\frown}(\sigma \upharpoonright [i+1, |\sigma|))$. We first check that R is well-founded on $L^{<\omega} \setminus \{0\}$. Suppose for a contradiction that $\langle \sigma_k : k \in \omega \rangle$ satisfies $\sigma_{k+1} R \sigma_k$, for each $k \in \omega$. Let $i_k < |\sigma_k|$ and η_k witness $\sigma_{k+1} R \sigma_k$. Define a finitely branching tree as follows. The kth level of the tree is the set $\{\langle k, j, \sigma_k(j) \rangle : j < |\sigma_k|\}$. For $j < i_k$, the sole immediate successor of $\langle k, j, \sigma_k(j) \rangle$ is $\langle k+1, j, \sigma_{k+1}(j) \rangle = \langle k+1, j, \sigma_k(j) \rangle$. The immediate successors of $\langle k, i_k, \sigma_k(i_k) \rangle$ are $\{\langle k+1, i_k+l, \eta_k(l) \rangle : l < |\eta_k|\}$. For $j \in [i_k + 1, |\sigma_k|)$, the sole immediate successor of $\langle k, j, \sigma_k(j) \rangle$ is $\langle k + 1, j + |\eta_k| - 1$ $1, \sigma_{k+1}(j+|\eta_k|-1)\rangle = \langle k+1, j+|\eta_k|-1, \sigma_k(j)\rangle$. For each $k \in \omega$, let us call $\langle k, i_k, \sigma_k(i_k) \rangle$ the active node at level k. Note that if $\langle k, j, x \rangle$ is a split node at level k, then it is active. Note also that if $\langle k, j, x \rangle$ is the active node at level k and if $\langle k+1, l, y \rangle$ is an immediate successor of it, then y < x. Now, choose a branch of this tree as follows. Choose a node at level 0 with infinitely many active nodes above it, and choose the first active node that is greater than or equal to it. There are infinitely many active nodes above this one, and hence infinitely many above one of its immediate successors. Choose such an immediate successor, and choose the first active node that is greater than or equal to this immediate successor. Continue in this fashion. There will be infinitely many active nodes along the branch thus chosen. Let $\{\langle k, j_k, x_k \rangle : k \in \omega\}$ be all the nodes on this branch. Now, it is clear that the set $\{x_k : k \in \omega\}$ does not have a minimal element, contradicting the well-foundedness of L.

Returning to the main thread of the argument, fix $n \in \omega$ such that X_n is uncountable. The hypothesis that there is no uncountable p.i.e. means that $L_{<\alpha}$ is countable for any $\alpha < \omega_1$. So choose $\sigma_\alpha \in L^{<\omega} \setminus \{0\}$ such that for any $x \in X_n \cap L_{<\alpha}$, there exists $i < |\sigma_\alpha|$ satisfying $x \le \sigma_\alpha(i)$, ensuring moreover that σ_α is *R*-minimal with respect to this property. Now, suppose $\alpha \le \beta < \omega_1$. We claim that for each $i < |\sigma_\alpha|$, there is a $j < \sigma_\beta$ such that $\sigma_\alpha(i) \le \sigma_\beta(j)$. Suppose not and fix $i < |\sigma_\alpha|$ so that for all $j < |\sigma_\beta|, \sigma_\alpha(i) \land \sigma_\beta(j) < \sigma_\alpha(i)$. Define $\eta \in L^{<\omega} \setminus \{0\}$ by $\eta(j) = \sigma_\alpha(i) \land \sigma_\beta(j)$, for each $j < |\sigma_\beta|$. Now, $\tau = (\sigma_\alpha \upharpoonright i)^{\frown}(\eta)^{\frown}(\sigma_\alpha \upharpoonright [i+1, |\sigma_\alpha|))$. Clearly, $\tau R \sigma_\alpha$ and yet, $\forall x \in X_n \cap L_{<\alpha} \exists k < |\tau| [x \le \tau(k)]$, contradicting the choice of σ_α .

Now pick an uncountable chain in L as follows. It follows from the above and the Δ -system lemma that there is a sequence $\langle F_{\alpha} : \alpha < \omega_1 \rangle$ such that

- (1) $F_{\alpha} \in [L]^{<\omega} \setminus \{0\}$
- (2) for each $\alpha, \beta \in \omega_1$, if $\alpha < \beta$, then $\max\{\operatorname{ht}(x) : x \in F_\alpha\} < \min\{\operatorname{ht}(x) : x \in F_\beta\}$
- (3) for each $\alpha, \beta \in \omega_1$, if $\alpha < \beta$, then $\forall x \in F_{\alpha} \exists y \in F_{\beta} [x < y]$.

Now, choose $x_{\alpha} \in F_{\alpha}$ such that $\forall \alpha < \beta < \omega_1 [x_{\alpha} < x_{\beta}]$ as follows. Given $x_{\alpha} \in F_{\alpha}$, use (3) to choose $x_{\alpha+1} \in F_{\alpha+1}$ with $x_{\alpha} < x_{\alpha+1}$. If α is a limit and $\{x_{\xi} : \xi < \alpha\}$ is given, then again by (3), there is $x_{\alpha} \in F_{\alpha}$ so that $\{\xi \in \alpha : x_{\xi} < x_{\alpha}\}$ is cofinal in α , whence $\forall \xi < \alpha [x_{\xi} < x_{\alpha}]$.

Corollary 13. Assume PID + $\mathfrak{p} > \omega_1$. Then there is no Suslin lower semi-lattice.

Theorem 14. Assume PID. Suppose there is a Suslin lower semi-lattice. Then there is $R \subset \mathcal{P}(\omega)$ such that $\langle R, \subsetneq \rangle$ is Suslin lower semi-lattice. Moreover, for any $x, y \in R, x \land y = x \cap y$.

Proof. Suppose $\langle L, < \rangle$ is Suslin. Without loss of generality, ht(*L*) = ω₁. By Theorem 12, there is a set *A* ∈ [*L*]^ω such that sups(*A*) = { $\bigvee B : B ⊂ A \land B$ has an upper bound in *L*} has size ω₁. Now, we find *R* as needed in *P*(*A*). Let *R* = {pred_A(*x*) : *x* ∈ sups(*A*)} ∪ {∅}. First, note that for each *x* ∈ sups(*A*), $x = \bigvee \text{pred}_A(x)$. Therefore, for any *x*, *y* ∈ sups(*A*), if $\text{pred}_A(x) ⊂ \text{pred}_A(y)$, then x ≤ y. On the other hand, it is clear that if x ≤ y, then $\text{pred}_A(x) ⊂ \text{pred}_A(y)$. Next, fix *x*, *y* ∈ sups(*A*) and note that $z = \bigvee (\text{pred}_A(x) ∩ \text{pred}_A(y))$ exists and is a member of sups(*A*). Note also that $x \land y$ is an upper bound for $\text{pred}_A(x) ∩ \text{pred}_A(y)$. So $z ≤ x \land y$. It follows that $\text{pred}_A(z) = \text{pred}_A(x) ∩ \text{pred}_A(y)$. So $\text{pred}_A(x) ∩ \text{pred}_A(y) ∈ R$ and therefore *R* is as needed.

There are two things worth noting about Theorem 14. First, there can be no $R \subset \mathcal{P}(\omega)$ such that $\langle R, \subsetneq \rangle$ is a Suslin tree. The easiest way to see this is to suppose that there is such an R and to force with $\langle R, \supset \rangle$. Since ω_1 is preserved, this adds a sequence $\langle x_{\alpha} : \alpha < \omega_1 \rangle$ such that $\forall \alpha < \beta < \omega_1 [x_{\alpha} \subsetneq x_{\beta}]$, which is impossible. So under PID, there must be Suslin lower semi-lattices that are very different from Suslin trees, provided of course that there are any at all.

Next, it is not possible to prove Theorem 14 in ZFC. In other words, Suslin lower semi-lattices may exist even if there are none inside $\mathcal{P}(\omega)$. This is because the OCA of Todorčević [9] implies that for any uncountable $R \subset \mathcal{P}(\omega)$, $\langle R, \subsetneq \rangle$ either contains an uncountable p.i.e. or an uncountable chain. Therefore, if $\langle R, \subsetneq \rangle$ is well-founded, then $\langle R, \subsetneq \rangle$ must contain an uncountable p.i.e. On the other hand, it is well-known that OCA is consistent with the existence of a Suslin tree, *a fortiori* with the existence of a Suslin lower semi-lattice.

4. A GENERIC SUSLIN LOWER SEMI-LATTICE IN $\mathcal{P}(\omega)$

In this section, we show that in contrast to Suslin trees, it is consistent to have a Suslin lower semi-lattice inside $\mathcal{P}(\omega)$. Therefore, Theorem 14 by itself does not tell us whether PID rules out the existence of Suslin lower semi-lattices. In fact, we can add by a c.c.c. forcing a Suslin lower semi-lattice in $\mathcal{P}(\omega)$ with some strong properties.

Definition 15. Let $\langle \mathbb{P}, \leq \rangle$ be a poset. For any $n \in \omega$ and $\sigma, \tau \in \mathbb{P}^n$, we write $\sigma \leq \tau$ to mean that $\forall i < n \ [\sigma(i) \leq \tau(i)]$. We say that $\langle \mathbb{P}, \leq \rangle$ is *powerfully c.c.c.* if for each $n \in \omega$, $\langle \mathbb{P}^n, \leq \rangle$ does not contain an uncountable p.i.e. In other words, for each $n \in \omega$ and for any $X \in [\mathbb{P}^n]^{\omega_1}$, $\exists \sigma, \tau \in X \ [\sigma \neq \tau \land \sigma \leq \tau]$.

This usage of the term "powerfully c.c.c." deviates somewhat from the standard usage. When applied to a poset, powerfully c.c.c. usually means that none of its finite powers has an uncountable family of pairwise incompatible elements – i.e. all

of the finite powers of that poset have the countable chain condition when considered as forcing notions. Since we are more concerned with uncountable p.i.e.s in this paper, we feel that this non-standard usage is justified. Having said that, we will be concerned with forcing notions having the countable chain condition both in this section and in Section 6. So when we say that a forcing notion has the c.c.c. we still mean as usual that it does not have any uncountable family of pairwise incompatible elements. This should cause no confusion.

The next theorem shows that there is a c.c.c. forcing notion of size \aleph_1 which not only adds $L \subset \mathcal{P}(\omega)$ such that $\langle L, \subsetneq \rangle$ is a Suslin lower semi-lattice, but also ensures that $\langle L, \subset \rangle$ is powerfully c.c.c. Again, this is in stark contrast with the situation for Suslin trees. It is to be expected that the existence of such an object can be proved from \Diamond . This is demonstrated in the next section, although the construction is more intricate than the corresponding construction of a Suslin tree.

In Section 6 we will require a ground model where there exist such $L \subset \mathcal{P}(\omega)$ and the GCH holds. The results in this section and the next one show that such a ground model exists.

Theorem 16. There is a c.c.c. poset of size ω_1 which adds $L \subset \mathcal{P}(\omega)$ such that

- (1) $\langle L, \subsetneq \rangle$ is a well-founded lower semi-lattice with $ht(L) = \omega_1$
- (2) for any $x, y \in L[x \land y = x \cap y]$
- (3) for each $n \in \omega$ for any $X \in [L^n]^{\omega_1}$, there exist $\sigma, \tau \in X$ such that $\sigma \neq \tau$ and $\forall i < n [\sigma(i) \subset \tau(i)].$

Moreover, if GCH holds in V, then GCH also holds in the extension.

Proof. Put $D = \{\langle 0, 0 \rangle\} \cup \{\langle \alpha, n \rangle : 1 \leq \alpha < \omega_1 \land n \in \omega\}$. \mathbb{P} is defined as follows. p is in \mathbb{P} iff $p = \langle F_p, \wedge_p, n_p, \sigma_p \rangle$ where

(4) $F_p \in [D]^{<\omega}$ with $\langle 0, 0 \rangle \in F_p$. $\langle F_p, <_p, \wedge_p \rangle$ is a lower semi-lattice, where for any $\langle \alpha, n \rangle, \langle \beta, m \rangle \in F_p$,

$$\left[\langle \alpha,n\rangle <_p \langle \beta,m\rangle\right] \iff \left[\langle \alpha,n\rangle \neq \langle \beta,m\rangle \text{ and } \langle \alpha,n\rangle = \langle \alpha,n\rangle \wedge_p \langle \beta,m\rangle\right].$$

- (5) for any $\langle \alpha, n \rangle, \langle \beta, m \rangle \in F_p$, if $\langle \alpha, n \rangle <_p \langle \beta, m \rangle$, then $\alpha < \beta$. In particular, for any $\langle \beta, m \rangle \in F_p$, $\langle 0, 0 \rangle \leq_p \langle \beta, m \rangle$. (6) $n_p \in \omega$ and $\sigma_p : F_p \to 2^{n_p}$
- (7) for any $\langle \alpha, n \rangle$, $\langle \beta, m \rangle$, and $\langle \gamma, l \rangle$ in F_p , if $\langle \alpha, n \rangle = \langle \beta, m \rangle \wedge_p \langle \gamma, l \rangle$, then $\forall i < n_p \left[\sigma_p(\langle \alpha, n \rangle)(i) = 1 \iff (\sigma_p(\langle \beta, m \rangle)(i) = 1 \text{ and } \sigma_p(\langle \gamma, l \rangle)(i) = 1) \right].$

For $p, q \in \mathbb{P}$, define $q \leq p$ to mean

- $\begin{array}{ll} (8) \ \ F_q \supset F_p \ \text{and} \ \wedge_q \upharpoonright (F_p \times F_p) = \wedge_p. \\ (9) \ \ n_q \ge n_p \ \text{and for each} \ \langle \alpha, n \rangle \in F_p, \ \sigma_q(\langle \alpha, n \rangle) \upharpoonright n_p = \sigma_p(\langle \alpha, n \rangle). \end{array}$

If G is a (\mathbf{V}, \mathbb{P}) -generic filter, then for each $\langle \alpha, n \rangle \in D$, put $x_{\alpha,n} = \{i \in \omega :$ $\exists p \in G[\langle \alpha, n \rangle \in F_p \land i < n_p \land \sigma_p(\langle \alpha, n \rangle)(i) = 1]\}. \text{ Put } L = \{x_{\alpha, n} : \langle \alpha, n \rangle \in D\}.$ Verifying that L satisfies (1) and (2) involves checking that several sets are dense. It is routine and we leave this to the reader. Note that for any $\langle \alpha, n \rangle, \langle \beta, m \rangle \in D$, if $\langle \alpha, n \rangle \neq \langle \beta, m \rangle$, then $x_{\alpha,n} \neq x_{\beta,m}$. Mark also that $L_0 = \{x_{0,0}\}$ and that for each $1 \le \alpha < \omega_1, \, L_\alpha = \{ x_{\alpha,n} : n \in \omega \}.$

The proof that (3) is satisfied is very similar to the proof that \mathbb{P} is c.c.c. So we will just check that (3) holds. Let \hat{L} be a \mathbb{P} name for L, and for each $\langle \alpha, n \rangle \in D$, let $\mathring{x}_{\langle \alpha,n \rangle}$ be a \mathbb{P} name for $x_{\alpha,n}$, where L and $x_{\alpha,n}$ are as defined above. Fix $n \in \omega$ and $X \in \mathbf{V}^{\mathbb{P}}$ such that $\Vdash X \subset L^n$. Suppose $p \in \mathbb{P}$ and $p \Vdash X$ is uncountable and $\forall \sigma, \tau \in \mathbb{P}$

 \dot{X} [if $\forall i < n [\sigma(i) \subset \tau(i)]$, then $\sigma = \tau$]. Find $\{p_{\alpha} : \alpha < \omega_1\} \subset \mathbb{P}$ and $\{s_{\alpha} : \alpha < \omega_1\}$ such that for each $\alpha < \omega_1, p_{\alpha} \leq p, s_{\alpha} \in D^n, s_{\alpha} \notin \{s_{\xi} : \xi < \alpha\}$, and $p_{\alpha} \Vdash \exists \sigma \in \dot{X} \forall i < n [\sigma(i) = \dot{x}_{s_{\alpha}(i)}]$. To make the notation easier, use $F_{\alpha}, \wedge_{\alpha}, n_{\alpha}$, and σ_{α} for $F_{p_{\alpha}}, \wedge_{p_{\alpha}}, n_{p_{\alpha}}$, and $\sigma_{p_{\alpha}}$ respectively. Now, we may assume that there exist $F \in [D]^{<\omega}$, and $u \subset i$ such that

- (10) $\{F_{\alpha} : \alpha < \omega_1\}$ forms a Δ -system with root F (so $\langle 0, 0 \rangle \in F$).
- (11) for all $\alpha < \beta < \omega_1, \forall \langle \gamma, k \rangle \in F \forall \langle \xi, m \rangle \in F_\alpha \setminus F \forall \langle \zeta, l \rangle \in F_\beta \setminus F [\gamma < \xi < \zeta].$
- (12) $\forall \alpha < \omega_1 \forall i < n \ [s_\alpha(i) \in F_\alpha]$. Moreover for each $\alpha < \omega_1$ and $i < n, \ s_\alpha(i) \in F \iff i \in u$
- (13) for any $\alpha < \beta < \omega_1$ there is a map $\pi_{\alpha,\beta} : F_\beta \to F_\alpha$ such that $\pi_{\alpha,\beta}$ is an isomorphism between $\langle F_\beta, <_\beta, \wedge_\beta \rangle$ and $\langle F_\alpha, <_\alpha, \wedge_\alpha \rangle$. Moreover, for any $\langle \gamma, k \rangle \in F, \pi_{\alpha,\beta}(\langle \gamma, k \rangle) = \langle \gamma, k \rangle$, for any $i < n, \pi_{\alpha,\beta}(s_\beta(i)) = s_\alpha(i)$, and for any $\langle \gamma, k \rangle \in F_\beta, \sigma_\beta(\langle \gamma, k \rangle) = \sigma_\alpha(\pi_{\alpha,\beta}(\langle \gamma, k \rangle))$. Note that this implies that $n_\alpha = n_\beta$.

Now, take $\alpha < \beta < \omega_1$. Define a condition q as follows. $F_q = F_\alpha \cup F_\beta$. $n_q = n_\alpha = n_\beta$. $\sigma_q = \sigma_\alpha \cup \sigma_\beta$. To get \wedge_q , extend $\wedge_\alpha \cup \wedge_\beta$ to all of F_q by stipulating that for any $\langle \xi, l \rangle \in F_\alpha \setminus F$ and any $\langle \zeta, j \rangle \in F_\beta \setminus F$, $\langle \xi, l \rangle \wedge_q \langle \zeta, j \rangle = \langle \xi, l \rangle \wedge_\alpha \pi_{\alpha,\beta}(\langle \zeta, j \rangle)$. It is not hard to verify that q is a condition and that $q \leq p_\alpha$ and $q \leq p_\beta$. Now, fix i < n. If $i \in u$, then $s_\alpha(i)$ and $s_\beta(i)$ are in F. Since $\pi_{\alpha,\beta}$ fixes everything in F, and since $\pi_{\alpha,\beta}(s_\beta(i)) = s_\alpha(i), s_\beta(i) = s_\alpha(i)$, and so $s_\alpha(i) \leq q s_\beta(i)$. If $i \notin u$, then $s_\alpha(i) \in F_\alpha \setminus F$ and $s_\beta(i) \in F_\beta \setminus F$. So $s_\alpha(i) \wedge_q s_\beta(i) = s_\alpha(i) \wedge_\alpha \pi_{\alpha,\beta}(s_\beta(i)) = s_\alpha(i) \wedge_\alpha s_\alpha(i) = s_\alpha(i)$. Therefore, in either case $s_\alpha(i) \leq q s_\beta(i)$. Now, let G be a (\mathbf{V}, \mathbb{P}) -generic filter with $q \in G$. In $\mathbf{V}[G]$, there exist $\sigma, \tau \in \mathring{X}[G]$ such that $\forall i < n [\sigma(i) = x_{s_\alpha(i)} \wedge \tau(i) = x_{s_\beta(i)}]$. Therefore, for each $i < n, \sigma(i) \subset \tau(i)$. On the other hand, there is i < n such that $s_\alpha(i) \neq s_\beta(i)$. For that $i, x_{s_\alpha(i)} \neq x_{s_\beta(i)}$. So, $\sigma \neq \tau$. But this contradicts our hypothesis about p.

5. \Diamond and a Suslin lower semi-lattice

In this section, we demonstrate a construction of a powerfully c.c.c. well-founded lower semi-lattice from \diamondsuit , which will be used in the next section. Let $L_{<\omega} := \mathsf{FT}^{<\aleph_0}$, where $\mathsf{FT}^{<\aleph_0}$ is the set of finite trees on ω , and for each $j \in \omega \setminus \{0\}$,

$$\begin{split} L_{[\omega \cdot j, \omega \cdot (j+1))} &:= \bigg\{ x \cup \{c \upharpoonright l; c \in F \And l \in \omega\} \, ; x \in L_{<\omega} \And F \in [\omega^{\omega}]^j \\ & \& \text{ every member of } F \text{ is eventually constant} \bigg\}. \end{split}$$

Then $L_{<\omega\cdot\omega}$ is a countable well-founded lower semi-lattice closed under (finite) intersections of height $\omega\cdot\omega$. Let $\langle\xi_{\alpha};\alpha\in\omega_1\rangle$ be the increasing enumeration of the set $[\omega\cdot\omega,\omega_1)\cap \text{Lim}$ (so $\xi_0 = \omega\cdot\omega, \xi_1 = \omega\cdot\omega + \omega$, etc). Let FT be the set of finite branching trees on ω . Let \mathcal{X} be a set, which plays a role of some kind of guessing sequence. In this section, \mathcal{X} will work as a \diamond -sequence in **Proposition 22** to our aim.

We will define forcing notions $\mathbb{P}(K)$ for countable subsets K of FT, which makes next ω many levels of well-founded lower semi-lattice, and we will build a sequence $\langle M_{\alpha}, G_{\alpha}; \alpha \in \omega_1 \rangle$ such that for each $\alpha \in \omega_1$,

• M_{α} is a countable elementary submodel of $H(\mathfrak{c}^+)$ with $\omega_1, \mathcal{X} \in M_{\alpha}$,

• G_{α} is an $(M_{\alpha}, \mathbb{P}(L_{<\xi_{\alpha}}))$ -generic (so G_{0} is an $(M_{0}, \mathbb{P}(L_{<\omega \cdot \omega}))$ -generic), and let $L_{[\xi_{\alpha},\xi_{\alpha+1})}$ be the countable subset of FT which comes from G_{α} ,

•
$$\langle M_{\beta}, G_{\beta}; \beta \in \alpha \rangle \in M_{\alpha}.$$

We will see below that L_{ω_1} is a powerfully c.c.c. well-founded lower semi-lattice closed under (finite) intersections if \mathcal{X} is a \diamond -sequence. More precise statement of **Conjecture 6** is that L_{ω_1} is a powerfully c.c.c. well-founded lower semi-lattice closed under (finite) intersections if \mathcal{X} is an enumeration of the set of reals of length ω_1 .

5.1. **Definitions.** For each $t \in \omega^{<\omega}$, let

$$\operatorname{cone}(t) := \left\{ s \in \omega^{<\omega}; t \subseteq s \right\},\$$

and define

$$\begin{aligned} \mathcal{T} &:= \left\{ \langle x, w, h \rangle \in \mathsf{FT} \times \left[\omega^{<\omega} \right]^{<\aleph_0} \times \omega; w \subseteq \omega^{\leq h} \right\}, \\ \mathcal{S} &:= \left\{ \langle x, w, h \rangle \in \mathcal{T}; \left(x \setminus \bigcup_{t \in w} \operatorname{cone}(t) \right) \cap \omega^h \neq \emptyset \right\}, \end{aligned}$$

and

$$\mathcal{S}^+ := \left\{ \langle x, w, h \rangle \in \mathcal{T}; \exists c \in \omega^{\omega} \text{ such that } \forall l \in \omega, c \upharpoonright l \in x \setminus \bigcup_{t \in w} \operatorname{cone}(t) \right\}.$$

For $\langle x, w, h \rangle$ and $\langle x', w', h' \rangle$ in \mathcal{T} , define the order on \mathcal{T} as follows:

$$\begin{split} \langle x,w,h\rangle \trianglelefteq \langle x',w',h'\rangle \ \text{if} \\ x \subseteq x' \ \& \ w \subseteq w' \ \& \ h \le h' \ \& \ x \cap \omega^{\le h} = x' \cap \omega^{\le h} \\ \& \ \forall s \in x' \setminus x, s \not\in \bigcup_{t \in w} \operatorname{cone}(t) \end{split}$$

For $\langle x, w, h \rangle \in \mathcal{T}$, define

$$x \upharpoonright_{w,h} := \left(x \cap \omega^{\leq h} \right) \cup \bigcup_{t \in w} \left(x \cap \operatorname{cone}(t) \right).$$

We note that for $\langle x, w, h \rangle$ and $\langle x', w', h' \rangle$ in \mathcal{T} , if $\langle x, w, h \rangle \leq \langle x', w', h' \rangle$, then

$$x\!\!\upharpoonright_{w,h} = x'\!\!\upharpoonright_{w,h}$$

We extend the order \trianglelefteq to the one on the product of n many copies of $L_{<\alpha}$ for each $n \in \omega$. For each $n \in \omega$, we let

$$\mathcal{S}^{n} := \left\{ \langle \overline{x}, w, h \rangle \in \mathsf{FT}^{n} \times \left[\omega^{<\omega} \right]^{<\aleph_{0}} \times \omega; \langle \overline{x}(k), w, h \rangle \in \mathcal{S} \text{ for every } k \in n \right\},\$$

and for each $\langle \overline{x}, w, h \rangle$ and $\langle \overline{x}', w', h' \rangle$ in \mathcal{S}^n , we define

$$\langle \overline{x}, w, h \rangle \leq^n \langle \overline{x}', w', h' \rangle$$
 if $\langle \overline{x}(k), w, h \rangle \leq \langle \overline{x}'(k), w', h' \rangle$ holds for every $k \in n$.

We fix an enumeration $\langle \langle w_i, h_i \rangle; i \in \omega \rangle$ of the set $[\omega^{<\omega}]^{<\aleph_0} \times \omega$. In the construction of the semi-lattice, for each $n \in \omega$ and $\alpha \in \omega_1$, we fix an enumeration $\langle \overline{x_{\alpha,j}}; j \in \omega \rangle$ of the set $(L_{\leq \alpha}{}^n) \setminus (L_{<\alpha}{}^n)$, and we add one more property for an elementary submodel M_{α} such that

• $\langle \langle w_i, h_i \rangle; i \in \omega \rangle$ and $\langle \overline{x_{\gamma,j}}; \gamma \in \alpha \& j \in \omega \rangle$ are members of M_{α} .

For $n \in \omega$, $\alpha \in \omega_1 + 1$ and a subset A of $L_{<\alpha}{}^n$, we assign a subset $\rho_{\alpha}(A)$ of $L_{<\alpha}{}^n$ as follows:

 $\rho_{\alpha}^{0,0,0}(A) := A.$ By recursion on $\gamma \in \alpha, i \in \omega$, and $j \in \omega$, we define

$$\rho_{\alpha}^{\gamma,i,j+1}(A) := \begin{cases} \rho_{\alpha}^{\gamma,i,j}(A) \cup \{\overline{\boldsymbol{x}_{\gamma,j}}\} & \text{if } \langle \overline{\boldsymbol{x}_{\gamma,j}}, w_i, h_i \rangle \in \mathcal{S} \text{ holds and for every} \\ \overline{a} \in \rho_{\alpha}^{\gamma,i,j}(A), \ \ ``\langle \overline{a}, w_i, h_i \rangle \notin \mathcal{S}^n \text{ holds}" \\ & \text{or } \ \ ``\langle \overline{a}, w_i, h_i \rangle \text{ and } \langle \overline{\boldsymbol{x}_{\gamma,j}}, w_i, h_i \rangle \text{ are} \\ & \text{incomparable with respect to } \leq^n " \\ \rho_{\alpha}^{\gamma,i,j}(A) & \text{otherwise,} \end{cases}$$

$$\gamma, i+1, 0$$
 (A) $\cdot = \begin{bmatrix} 1 & \gamma, i, j \end{bmatrix}$

$$\rho_{\alpha}^{\gamma,i+1,0}(A) := \bigcup_{j \in \omega} \rho_{\alpha}^{\gamma,i,j}(A),$$

and

$$\rho^{\gamma,0,0}_{\alpha}(A) := \bigcup_{\beta < \gamma, \ i,j \in \omega} \rho^{\beta,i,j}_{\alpha}(A),$$

and define

$$\rho_{\alpha}(A) := \bigcup_{\gamma \in \alpha, \ i, j \in \omega} \rho_{\alpha}^{\gamma, i, j}(A).$$

We note that for each $n \in \omega$,

- if A is a p.i.e. in $L_{\leq\alpha}{}^n$ (i.e. for any distinct \overline{a} and \overline{b} in A, there is $k \in n$ such that $\overline{a}(k)$ and $\overline{b}(k)$ are incomparable with respect to \subseteq), then for every $\langle w, h \rangle \in [\omega^{\leq \omega}]^{\leq \aleph_0} \times \omega$ and $\overline{a}, \overline{b} \in \rho_{\alpha}$, whenever both $\langle \overline{a}, w, h \rangle$ and $\langle \overline{b}, w, h \rangle$ are in S^n , they are incomparable with respect to \leq^n ,
- if A is a subset of $L_{<\alpha}{}^n$, then for every $\overline{b} \in L_{<\alpha}{}^n$ and $\langle w, h \rangle \in [\omega^{<\omega}]^{<\aleph_0} \times \omega$ with $\langle \overline{b}, w, h \rangle \in S^n$, there exists $\overline{a} \in \rho_{\alpha}(A)$ such that $\langle \overline{a}, w, h \rangle$ and $\langle \overline{b}, w, h \rangle$ are comparable with respect to \leq^n ,
- if A is a subset of $L_{<\omega_1}$, and M is a countable elementary submodel of $H(\mathfrak{c}^+)$ so that $A, \langle \langle w_i, h_i \rangle; i \in \omega \rangle, \langle L_{\alpha}; \alpha \in \omega_1 \rangle$ and $\langle \overline{x_{\alpha,j}}; \alpha \in \omega_1 \& j \in \omega \rangle$ are members of M, then

$$\rho_{\omega_1 \cap M}(A \cap M) = \rho_{\omega_1 \cap M}(A) \cap M.$$

For each $n \in \omega$ and $\alpha \in \omega_1 + 1$, we define that a subset A of $L_{<\alpha}{}^n$ is called \trianglelefteq^n -mc in $L_{<\alpha}{}^n$ if there exists a maximal p.i.e. A' in $L_{<\alpha}{}^n$ such that $A = \rho_{\alpha}(A')$. We note that for each $n \in \omega$, if a well-founded semi-lattice has no uncountable subsets of $L_{<\omega_1}{}^n$ which are \trianglelefteq^n -mc in $L_{<\omega_1}{}^n$, then it has no uncountable p.i.e. in $L_{<\omega_1}{}^n$.

When we build $L_{<\alpha}$ for some countable limit ordinal α , we build members $\boldsymbol{x}_{\langle i,j\rangle}^G$ of FT for all $i,j \in \omega$ as the rank $\alpha + j$ -th member of the set $L_{<\alpha+\omega}$, by the forcing notion $\mathbb{P}(L_{<\alpha})$. To work our desired density arguments, we build $L_{<\alpha}$ which satisfies the following property $\mathsf{A}(L_{<\alpha})$:

For every
$$\langle x, w, h \rangle \in \left(L_{<\alpha} \times [\omega^{<\omega}]^{<\aleph_0} \times \omega \right) \cap \mathcal{S}^+, \ s \in \omega^{>h}$$
 with $s \upharpoonright h \in x \setminus \bigcup_{t \in w} \operatorname{cone}(t), \ y \in L_{<\alpha}$ with $y \upharpoonright_{w,h} \subseteq x \upharpoonright_{w,h}$, and $\gamma \in \alpha$,

10

there exists $z \in L_{<\alpha}$ such that

$$\begin{split} \langle x, w, h \rangle \trianglelefteq \langle z, w, h \rangle & \& \ y \subseteq z \ \& \ \operatorname{ht}(z) \ge \gamma \\ & \& \ z \cap \omega^{\le |s|} = \left((x \cup y) \cap \omega^{\le |s|} \right) \cup \{s\} \\ & \& \ \exists c \in \omega^{\omega} \ \text{such that} \ s \subseteq c \ \text{and} \ \forall l \in \omega, c \upharpoonright l \in z. \end{split}$$

We recall that $L_{<\omega} := \mathsf{FT} \cap [\omega^{<\omega}]^{<\aleph_0}$, and for each $j \in \omega \setminus \{0\}$,

$$L_{[\omega \cdot j, \omega \cdot (j+1))} := \left\{ x \cup \{c \upharpoonright l; c \in F \& l \in \omega\} ; x \in L_{<\omega} \& F \in [\omega^{\omega}]^j \right\}$$

& every member of F is eventually constant $\Big\}$.

Then $L_{\langle \omega \cdot \omega \rangle}$ is a countable well-founded lower semi-lattice closed under (finite) intersections of height $\omega \cdot \omega$, which satisfies $A(L_{\langle \omega \cdot \omega \rangle})$.

In the next paragraph, we will define the forcing notion $\mathbb{P}(L_{<\alpha})$ which consists of the tuples $\langle \langle x_{\sigma}^{p}; \sigma \in m^{p} \times n^{p} \rangle, w^{p}, h^{p}, f^{p} \rangle$. Each x_{σ}^{p} is a non-decreasing sequence of members of $L_{<\alpha}$, which is a approximation of \mathbf{x}_{σ}^{G} . f^{p} is a finite function which assigns the role of z for each pair x and y of members of $L_{<\alpha+\omega}$ in the property $\mathsf{A}(L_{<\alpha+\omega})$. The requirement **5** in the definition of $\mathbb{P}(L_{<\alpha})$ works for $\mathbb{P}(L_{<\alpha+\omega})$ to be closed under finite intersection (Proposition 20).

The forcing notion $\mathbb{P}(L_{<\alpha})$ consists of the tuples $\langle\langle x_{\sigma}^{p}; \sigma \in m^{p} \times n^{p} \rangle, w^{p}, h^{p}, f^{p} \rangle$ such that

1: there exists $l^p \in \omega$ such that for every $\sigma \in m^p \times n^p$, x^p_{σ} is a sequence of members of $L_{<\alpha}$ of length l^p such that for every $k < l^p - 1$, $x^p_{\sigma}(k) \subseteq x^p_{\sigma}(k+1)$,

2: for every $\sigma \in m^p \times n^p$, $\langle x^p_{\sigma}(l^p - 1), w^p, h^p \rangle \in \mathcal{S}^+$,

- **3:** for every $\langle i,j \rangle \in m^p \times (n^p 1)$ and $k \in l^p, x^p_{\langle i,j \rangle}(k) \subseteq x^p_{\langle i,j+1 \rangle}(k)$,
- **4:** f^p is a finite partial function from the set

 $\left((L_{<\alpha} \times (m^p \times n^p)) \cup ((m^p \times n^p) \times L_{<\alpha}) \cup (m^p \times n^p)^2 \right) \times \left[\omega^{<\omega} \right]^{<\aleph_0} \times \omega \times \omega^{<\omega} \times \omega$

into $m^p \times n^p$ such that

• for every $\langle \langle y, \sigma \rangle, w, h, s, j \rangle$ in the set

$$dom(f^p) \cap \left((L_{<\alpha} \times (m^p \times n^p)) \times \left[\omega^{<\omega} \right]^{<\aleph_0} \times \omega \times \omega^{<\omega} \times \omega \right),$$

$$- w \subseteq w^p \cap \omega^{\le h}, h \le h^p, \text{ and } \langle x^p_{\sigma}(l^p - 1), w, h \rangle \in \mathcal{S}^+,$$

$$- s \in \omega^{>h} \text{ with}$$

$$s \upharpoonright h \in x^p_{\sigma}(l^p - 1) \setminus \bigcup_{t \in w} \text{ cone}(t)$$

and $|s| \le h^p$

and
$$|s| \leq h^{p}$$
,
 $= y \upharpoonright_{w,h} \subseteq x_{\sigma}^{p}(l^{p}-1) \upharpoonright_{w,h}$,
 $= (2nd \text{ cor. of } f^{p}(\langle y, \sigma \rangle, w, h, s, j))$
 $\geq \max\{j, (2nd \text{ cor. of } \sigma) + 1\},$
 $= \text{ for every } k \in l^{p}$,
 $\langle x_{\sigma}^{p}(k), w, h \rangle \trianglelefteq \langle x_{f^{p}(\langle y, \sigma \rangle, w, h, s, j)}^{p}(k), w, h \rangle$

and

$$y \subseteq x_{f^p(\langle y,\sigma\rangle,w,h,s,j)}^p(k),$$

$$x_{f^p(\langle y,\sigma\rangle,w,h,s,j)}^p(l^p-1)\cap\omega^{\leq |s|} = \left((x_{\sigma}^p(l^p-1)\cup y)\cap\omega^{\leq |s|}\right)\cup\{s\}$$

and

 $\begin{aligned} \exists c \in \omega^{\omega} \text{ such that } s \subseteq c \text{ and } \forall l \in \omega, c \upharpoonright l \in x_{f^{p}(\langle y, \sigma \rangle, w, h, s, j)}^{p}(l^{p} - 1), \\ \bullet \text{ for every } \langle \langle \sigma, y \rangle, w, h, s, j \rangle \text{ in the set} \\ \operatorname{dom}(f^{p}) \cap \left(\left((m^{p} \times n^{p}) \times L_{<\alpha} \right) \times \left[\omega^{<\omega} \right]^{<\aleph_{0}} \times \omega \times \omega^{<\omega} \times \omega \right), \\ &- w \subseteq w^{p} \cap \omega^{\leq h}, h \leq h^{p} \text{ and } \langle y, w, h \rangle \in \mathcal{S}^{+}, \\ &- s \in \omega^{>h} \text{ with} \\ &s \upharpoonright h \in y \setminus \bigcup_{t \in w} \operatorname{cone}(t) \\ &\text{ and } |s| \leq h^{p}, \end{aligned}$

and
$$|s| \leq h^{r}$$
,
 $-x_{\sigma}^{p}(l^{p}-1) \upharpoonright_{w,h} \subseteq y \upharpoonright_{w,h}$,
 $-(2nd \text{ cor. of } f^{p}(\langle \sigma, y \rangle, w, h, s, j))$
 $\geq \max\{j, (2nd \text{ cor. of } \sigma) + 1\},$

$$-$$
 for every $k \in l^p$,

$$\langle y,w,h\rangle \trianglelefteq \left\langle x^p_{f^p(\langle \sigma,y\rangle,w,h,s,j)}(k),w,h\right\rangle$$

and

$$x^p_{\sigma}(k) \subseteq x^p_{f^p(\langle \sigma, y \rangle, w, h, s, j)}(k),$$

$$\begin{aligned} x^p_{f^p(\langle \sigma, y \rangle, w, h, s, j)}(l^p - 1) \cap \omega^{\leq |s|} &= \left((y \cup x^p_{\sigma}(l^p - 1)) \cap \omega^{\leq |s|} \right) \cup \{s\} \\ \text{and} \end{aligned}$$

 $\exists c \in \omega^{\omega} \text{ such that } s \subseteq c \text{ and } \forall l \in \omega, c \upharpoonright l \in x_{f^{p}(\langle \sigma, y \rangle, w, h, s, j)}^{p}(l^{p} - 1),$

• for every $\langle \langle \sigma, \tau \rangle, w, h, s, j \rangle$ in the set $\operatorname{dom}(f^p) \cap \left(\left((m^p \times n^p)^2 \right) \times \left[\omega^{<\omega} \right]^{<\aleph_0} \times \omega \times \omega^{<\omega} \times \omega \right),$ $- w \subseteq w^p \cap \omega^{\leq h}, h \leq h^p, \text{ and } \langle x^p_{\sigma}(l^p - 1), w, h \rangle \in \mathcal{S}^+,$ $- s \in \omega^{>h} \text{ with}$ $s \upharpoonright h \in x^p_{\sigma}(l^p - 1) \setminus \bigcup_{t \in w} \operatorname{cone}(t)$ and $|s| \leq h^p,$ $- x^p_{\tau}(l^p - 1) \upharpoonright_{w,h} \subseteq x^p_{\sigma}(l^p - 1) \upharpoonright_{w,h},$

(2nd cor. of $f^p(\langle \sigma, \tau \rangle, w, h, s, j))$

 $\geq \max\left\{j, (2nd \text{ cor. of } \sigma) + 1, (2nd \text{ cor. of } \tau) + 1\right\},\$

- for every
$$k \in l^p$$
,
 $\langle x^p_{\sigma}(k), w, h \rangle \leq \left\langle x^p_{f^p(\langle \sigma, \tau \rangle, w, h, s, j)}(k), w, h \right\rangle$
and
 $x^p_{\tau}(k) \subseteq x^p_{f^p(\langle \sigma, \tau \rangle, w, h, s, j)}(k),$

а

$$x^p_{\tau}(k) \subseteq x^p_{f^p(\langle \sigma, \tau \rangle, w, h, s, j)}(k)$$

$$\begin{aligned} x^p_{f^p(\langle \sigma, \tau \rangle, w, h, s, j)}(l^p - 1) \cap \omega^{\leq |s|} &= \left((x^p_{\sigma}(l^p - 1) \cup x^p_{\tau}(l^p - 1)) \cap \omega^{\leq |s|} \right) \cup \{s\} \\ \text{and} \end{aligned}$$

 $\exists c \in \omega^{\omega}$ such that $s \subseteq c$ and $\forall l \in \omega, c \upharpoonright l \in x_{f^{p}(\langle \sigma, \tau \rangle, w, h, s, j)}^{p}(l^{p}-1),$

- for any $\langle \langle \sigma, y \rangle, w, h, s, j \rangle$, $\langle \langle y, \sigma \rangle, w, h, s, j \rangle$ and $\langle \langle \sigma, \tau \rangle, w, h, s, j \rangle$ in $\operatorname{dom}(f^p),$
 - (1st cor. of $f^p(\langle \sigma, y \rangle, w, h, s, j)) \neq$ (1st cor. of σ), (1st cor. of $f^p(\langle y, \sigma \rangle, w, h, s, j)) \neq (1st \text{ cor. of } \sigma)$,
 - (1st cor. of $f^p(\langle \sigma, \tau \rangle, w, h, s, j)) \neq$ (1st cor. of σ),
- (1st cor. of $f^p(\langle \sigma, \tau \rangle, w, h, s, j)) \neq$ (1st cor. of τ),
- for any distinct $\langle \Sigma, w, h, s, j \rangle$ and $\langle \Sigma', w', h', s', j' \rangle$ in dom (f^p) ,
- (1st cor. of $f^p(\Sigma, w, h, s, j)$) \neq (1st cor. of $f^p(\Sigma', w', h', s', j')$),
- **5:** for every $\langle i, j \rangle$ and $\langle i', j' \rangle$ in $m^p \times n^p$ with $i \neq i'$,

$$\left\langle x^p_{\langle i,j\rangle}(l^p-1)\cap x^p_{\langle i',j'\rangle}(l^p-1), w^p, h^p\right\rangle \notin \mathcal{S},$$

unless there are $y \in L_{<\alpha}$, $\tau \in m^p \times n^p$, $j_0 \ge j$, $j'_0 \ge j'$, $j_1 \le j$, $j'_1 \le j'$, $w \in [\omega^{<\omega}]^{<\aleph_0}, h, k \text{ and } j' \text{ in } \omega \text{ and } s \in \omega^{<\omega} \text{ such that}$

• $\langle \langle y, \langle i, j_0 \rangle \rangle, w, h, s, k \rangle \in \text{dom}(f^p)$ and

$$f^{p}(\langle y, \langle i, j_{0} \rangle \rangle, w, h, s, k) = \langle i', j_{1}' \rangle,$$

or

• $\langle \langle \langle i, j_0 \rangle, y \rangle, w, h, s, k \rangle \in \text{dom}(f^p)$ and $f^{p}(\langle\langle i, j_{0}\rangle, y\rangle, w, h, s, k) = \langle i', j_{1}'\rangle,$

or

or

- $\langle \langle \tau, \langle i, j_0 \rangle \rangle, w, h, s, k \rangle \in \operatorname{dom}(f^p)$ and $f^p(\langle \tau, \langle i, j_0 \rangle \rangle, w, h, s, k) = \langle i', j_1' \rangle,$
- $\langle \langle \langle i, j_0 \rangle, \tau \rangle, w, h, s, k \rangle \in \text{dom}(f^p)$ and $f^{p}(\langle\langle i, j_{0}\rangle, \tau\rangle, w, h, s, k) = \langle i', j'_{1}\rangle,$

or

• $\langle \langle y, \langle i', j'_0 \rangle \rangle, w, h, s, k \rangle \in \text{dom}(f^p)$ and $f^{p}(\langle y, \langle i', j_{0}' \rangle \rangle, w, h, s, k) = \langle i, j_{1} \rangle,$

or

or

• $\langle \langle \langle i', j'_0 \rangle, y \rangle, w, h, s, k \rangle \in \text{dom}(f^p)$ and $f^{p}(\langle \langle i', j_{0}' \rangle, y \rangle, w, h, s, k) = \langle i, j_{1} \rangle,$

•
$$\langle \langle \tau, \langle i', j_0 \rangle \rangle, w, h, s, k \rangle \in \text{dom}(f^p) \text{ and}$$

 $f^p(\langle \tau, \langle i', j'_0 \rangle \rangle, w, h, s, k) = \langle i, j_1 \rangle,$

• $\langle \langle \langle i', j'_0 \rangle, \tau \rangle, w, h, s, k \rangle \in \text{dom}(f^p)$ and

$$f^{p}(\langle\langle i', j_{0}'\rangle, \tau\rangle, w, h, s, k) = \langle i, j_{1}\rangle.$$

For conditions p and q in $\mathbb{P}(L_{<\alpha})$, define that $q \leq_{\mathbb{P}(L_{<\alpha})} p$ if

- $m^q \ge m^p, n^q \ge n^p, w^q \supseteq w^p, h^q \ge h^p, f^q \supseteq f^p,$
- for every $\sigma \in m^p \times n^p$, x^q_σ is an end-extension of x^p_σ ,
- for every $\sigma \in m^p \times n^p$ and $k \in l^q \setminus l^p$,

$$\langle x^p_{\sigma}(l^p-1), w^p, h^p \rangle \leq \langle x^q_{\sigma}(k), w^q, h^q \rangle,$$

• for every $\sigma \in \operatorname{ran}(f^q) \setminus \operatorname{ran}(f^p)$, (1st cor. of σ) $\in m^q \setminus m^p$.

For a $\mathbb{P}(L_{\leq \alpha})$ -generic G and $\sigma \in \omega \times \omega$, we define

$$\boldsymbol{x}_{\sigma}^{G} := \bigcup \left\{ \boldsymbol{x}_{\sigma}^{p}(l); p \in G, \sigma \in m^{p} \times n^{p}, \text{ and } l < l^{p} \right\}.$$

We note here that for any $p \in \mathbb{P}(L_{<\alpha})$ and $i, i' \in \omega$, if both i and i' are in m^p , then p decides whether $\boldsymbol{x}_{\langle i,j \rangle}^G$ is comparable with $\boldsymbol{x}_{\langle i',j' \rangle}^G$ for every $j, j' \in \omega$. In fact, for each $\langle i,j \rangle$ and $\langle i',j' \rangle$ in $m^p \times n^p$,

- if i = i', then $\boldsymbol{x}_{\langle i,j \rangle}^G \subseteq \boldsymbol{x}_{\langle i,j' \rangle}^G$ iff $j \leq j'$,
- if $i \neq i'$ and one of the unless-cases in the requirement 5 happens, then $\boldsymbol{x}_{\langle i,j \rangle}^{G}$ and $\boldsymbol{x}_{\langle i,j \rangle}^{G}$ are comparable, and
- if $i \neq i'$ and any of the unless-cases in the requirement 5 doesn't happen, then $x_{(i,j)}^G$ and $x_{(i,j')}^G$ are incomparable and moreover,

$$\boldsymbol{x}^{G}_{\langle i,j\rangle} \cap \boldsymbol{x}^{G}_{\langle i,j'\rangle} = x^{p}_{\langle i,j\rangle}(l^{p}-1) \cap x^{p}_{\langle i',j'\rangle}(l^{p}-1).$$

5.2. Density arguments.

Lemma 17. If $p \in \mathbb{P}(L_{<\alpha})$, $w \in [\omega^{<\omega}]^{<\aleph_0}$ and $h \in \omega$ with $w \subseteq \omega^{\leq h}$ such that for every $\sigma \in m^p \times n^p$, $\langle x_{\sigma}^p(l^p - 1), w, h \rangle \in \mathcal{S}$, $\langle z_{\sigma}; \sigma \in m^p \times n^p \rangle$ is a sequence of members of $L_{<\alpha}$ such that for every $\sigma \in m^p \times n^p$,

$$\langle x^p_{\sigma}(l^p-1), w^p, h^p \rangle \trianglelefteq \langle z_{\sigma}, w^p, h^p \rangle \text{ and } \langle z_{\sigma}, w^p, h^p \rangle \in \mathcal{S}^+,$$

and $y \in L_{<\alpha}$ with $\langle y, w^p, h^p \rangle \in S^+$ and $\langle y, w, h \rangle \in S$, then there exists $q \in \mathbb{P}(L_{<\alpha})$ such that $q \leq_{\mathbb{P}(L_{<\alpha})} p, m^q \geq m^p + 1, l^q = l^p + 1, w^q \supseteq w$, and for every $\sigma \in m^p \times n^p$,

$$\langle z_{\sigma}(l^p), w^p, h^p \rangle \leq \langle x_{\sigma}^q(l^p), w^p, h^p \rangle,$$

 $and \ \langle y, w^p, h^p \rangle \trianglelefteq \Big\langle x^q_{\langle m^p, n^p-1 \rangle}(l^p), w^p, h^p \Big\rangle.$

Proof. Suppose that $p \in \mathbb{P}(L_{<\alpha}), w \in [\omega^{<\omega}]^{<\aleph_0}, \langle z_{\sigma}; \sigma \in m^p \times n^p \rangle$ is a sequence of members of $L_{<\alpha}$ such that for every $\sigma \in m^p \times n^p$,

$$\langle x^p_{\sigma}(l^p-1), w^p, h^p \rangle \leq \langle z_{\sigma}, w^p, h^p \rangle \text{ and } \langle z_{\sigma}, w^p, h^p \rangle \in \mathcal{S}^+,$$

and $y \in L_{\leq \alpha}$ with $\langle y, w^p, h^p \rangle \in \mathcal{S}^+$. We will make an extension q of p as desired.

We let $l^q := l^p + 1$, $m^q := m^p + 1$, $n^q := n^p$. and take $h^q \in \omega$ which is larger than both h^p and h such that $w \subseteq \omega^{\leq h^q}$. To simplify notation, for every $j \in n^p$, we define a non-decreasing sequence $x^p_{\langle m^p, j \rangle}$ of members of $L_{<\alpha}$ of length l^p such that for each $k \in l^p$, $x^p_{\langle m^p, j \rangle}(k) := y$, and let $z_{\langle m^p, j \rangle} := y$. We notice that the tuple $\langle \langle x^p_{\sigma}; \sigma \in (m^p + 1) \times n^p \rangle, w^p, h^p, f^p \rangle$ may not be a condition in $\mathbb{P}(L_{<\alpha})$, because it may not satisfy the requirement **5**.

We take a sequence $\langle s_{\sigma}; \sigma \in (m^p + 1) \times n^p \rangle$ of members of ω^{h^q} such that

• for each $\sigma \in (m^p + 1) \times n^p$,

$$s_{\sigma} \upharpoonright h^p \in z_{\sigma} \setminus \bigcup_{t \in w^p} \operatorname{cone}(t),$$

- if σ and τ are different elements of $(m^p + 1) \times n^p$, then $s_{\sigma} \neq s_{\tau}$,
- for each $\sigma \in (m^p + 1) \times n^p$,

$$s_{\sigma} \notin \left(\bigcup_{\tau \in (m^p+1) \times n^p} z_{\tau}\right) \cup \bigcup_{t \in w} \operatorname{cone}(t).$$

By induction on $j \in n^p$, we choose $x^q_{\langle i,j \rangle}(l^p) \in L_{<\alpha}$ such that

• for every $i \in (m^p + 1)$,

$$x^{q}_{\langle i,j-1\rangle}(l^{p}) \subseteq x^{q}_{\langle i,j\rangle}(l^{p}) \text{ and } \left\langle z_{\langle i,j\rangle}, w^{p}, h^{p} \right\rangle \trianglelefteq \left\langle x^{q}_{\langle i,j\rangle}(l^{p}), w^{p}, h^{p} \right\rangle,$$

$$\left\langle x^{q}_{\langle i,j\rangle}(l^{p}), w^{p}, h^{p} \right\rangle \in \mathcal{S}^{+}$$

• if there exists $\langle \langle \sigma', y' \rangle, w', h', s', j' \rangle \in \operatorname{dom}(f^p)$ such that $f^p(\langle \sigma', y' \rangle, w', h', s', j') = \langle i, j \rangle,$

then

$$y' \subseteq x^q_{\langle i,j \rangle}(l^p), \quad \langle x^q_{\sigma'}(l^p), w', h' \rangle \trianglelefteq \left\langle x^q_{\langle i,j \rangle}(l^p), w', h' \right\rangle,$$

and

$$x^{q}_{\langle i,j\rangle}(l^{p})\cap\omega^{\leq h^{q}} = \left(\left(y'\cup x^{q}_{\sigma'}(l^{p})\cup z_{\langle i,j\rangle}\right)\cap\omega^{\leq h^{q}}\right)\cup\{s_{\langle i,j\rangle}\}$$

• if there exists $\langle \langle y', \sigma' \rangle, w', h', s', j' \rangle \in \operatorname{dom}(f^p)$ such that $f^p(\langle y', \sigma' \rangle, w', h', s', j') = \langle i, j \rangle,$

then

$$x_{\sigma'}^q(l^p) \subseteq x_{\langle i,j \rangle}^q(l^p), \quad \langle y', w', h' \rangle \trianglelefteq \left\langle x_{\langle i,j \rangle}^q(l^p), w', h' \right\rangle,$$

and

$$x^{q}_{\langle i,j\rangle}(l^{p})\cap\omega^{\leq h^{q}} = \left(\left(x^{q}_{\sigma'}(l^{p})\cup y'\cup z_{\langle i,j\rangle}\right)\cap\omega^{\leq h^{q}}\right)\cup\{s_{\langle i,j\rangle}\}$$

• if there exists $\langle \langle \sigma', \tau' \rangle, w', h', s', j' \rangle \in \operatorname{dom}(f^p)$ such that $f^p(\langle \sigma', \tau' \rangle, w', h', s', j') = \langle i, j \rangle,$

then

$$x^{q}_{\sigma'}(l^{p}) \subseteq x^{q}_{\langle i,j \rangle}(l^{p}), \quad \langle x^{q}_{\tau}(l^{p}), w', h' \rangle \leq \left\langle x^{q}_{\langle i,j \rangle}(l^{p}), w', h' \right\rangle,$$

and

$$x^q_{\langle i,j\rangle}(l^p)\cap\omega^{\leq h^q} = \left(\left(x^q_{\sigma'}(l^p)\cup x^q_{\tau'}(l^p)\cup z_{\langle i,j\rangle}\right)\cap\omega^{\leq h^q}\right)\cup\{s_{\langle i,j\rangle}\},$$

• otherwise,

$$x^{q}_{\langle i,j\rangle}(l^{p})\cap\omega^{\leq h^{q}}=\left(z_{\langle i,j\rangle}\cap\omega^{\leq h^{q}}\right)\cup\{s_{\langle i,j\rangle}\}.$$

This can be done by the property $\mathsf{A}(L_{<\alpha})$. We demonstrate the case that there exists $\langle\langle \sigma', \tau' \rangle, w', h', s', j' \rangle \in \operatorname{dom}(f^p)$ such that $f^p(\langle \sigma', \tau' \rangle, w', h', s', j') = \langle i, j \rangle$. By the inductive construction, $x^q_{\sigma'}(l^p)$ and $x^q_{\tau'}(l^p)$ have already been defined in this situation. It is true that $x^p_{\sigma'}(l^p-1) \subseteq x^p_{\langle i,j \rangle}(l^p-1)$ and $x^q_{\tau'}(l^p-1) \subseteq x^p_{\langle i,j \rangle}(l^p-1)$. Moreover, since

$$\langle x_{\sigma'}^p(l^p-1), w^p, h^p \rangle \leq \langle z_{\sigma'}, w^p, h^p \rangle \leq \langle x_{\sigma'}^q(l^p), w^p, h^p \rangle$$

and

$$\langle x^p_{\tau'}(l^p-1), w^p, h^p \rangle \trianglelefteq \langle z_{\tau'}, w^p, h^p \rangle \trianglelefteq \langle x^q_{\tau'}(l^p), w^p, h^p \rangle,$$

it is also true that

$$x_{\sigma'}^q(l^p)\!\upharpoonright_{w^p,h^p} = x_{\sigma'}^p(l^p-1)\!\upharpoonright_{w^p,h^p} \subseteq x_{\langle i,j\rangle}^p(l^p-1)\!\upharpoonright_{w^p,h^p} = z_{\langle i,j\rangle}\!\upharpoonright_{w^p,h^p}$$

and

$$x^q_{\tau'}(l^p)\!\upharpoonright_{w^p,h^p} = x^p_{\tau'}(l^p-1)\!\upharpoonright_{w^p,h^p} \subseteq x^p_{\langle i,j\rangle}(l^p-1)\!\upharpoonright_{w^p,h^p} = z_{\langle i,j\rangle}\!\upharpoonright_{w^p,h^p}$$

Since $x^p_{\langle i,j-1\rangle}(l^p-1) \subseteq x^p_{\langle i,j\rangle}(l^p-1) \subseteq z_{\langle i,j\rangle}$ and

$$\left\langle x^p_{\langle i,j-1\rangle}(l^p-1), w^p, h^p \right\rangle \trianglelefteq \left\langle z_{\langle i,j-1\rangle}, w^p, h^p \right\rangle \trianglelefteq \left\langle x^q_{\langle i,j-1\rangle}(l^p), w^p, h^p \right\rangle,$$

we note that

$$x^q_{\langle i,j-1\rangle}(l^p)\!\upharpoonright_{w^p,h^p} = x^p_{\langle i,j-1\rangle}(l^p-1)\!\upharpoonright_{w^p,h^p} \subseteq z_{\langle i,j\rangle}\!\upharpoonright_{w^p,h^p}.$$

So by applying the property $\mathsf{A}(L_{<\alpha})$ four times with $\langle w^p, h^p, s_{\langle i,j \rangle} \rangle$ repeatedly, we can take $x^q_{\langle i,j \rangle}(l^p) \in L_{<\alpha}$ such that

$$\begin{split} & \left\langle x^{q}_{\sigma'}(l^{p}), w^{p}, h^{p} \right\rangle \trianglelefteq \left\langle x^{q}_{\langle i,j \rangle}(l^{p}), w^{p}, h^{p} \right\rangle, \quad \left\langle x^{q}_{\tau'}(l^{p}), w^{p}, h^{p} \right\rangle \trianglelefteq \left\langle x^{q}_{\langle i,j \rangle}(l^{p}), w^{p}, h^{p} \right\rangle, \\ & \left\langle x^{q}_{\langle i,j-1 \rangle}(l^{p}), w^{p}, h^{p} \right\rangle \trianglelefteq \left\langle x^{q}_{\langle i,j \rangle}(l^{p}), w^{p}, h^{p} \right\rangle, \quad \left\langle z_{\langle i,j \rangle}, w^{p}, h^{p} \right\rangle \trianglelefteq \left\langle x^{q}_{\langle i,j \rangle}(l^{p}), w^{p}, h^{p} \right\rangle, \end{split}$$

and

$$x^{q}_{\langle i,j\rangle}(l^{p})\cap\omega^{\leq h^{q}} = \left(\left(x^{q}_{\sigma'}(l^{p})\cup x^{q}_{\tau'}(l^{p})\cup z_{\langle i,j\rangle}\right)\cap\omega^{\leq h^{q}}\right)\cup\{s_{\langle i,j\rangle}\}.$$

Then we note that for each $\langle i, j \rangle \in m^p \times n^p$,

• if there is $\langle \langle \langle i', j' \rangle, y' \rangle, w', h', s', k' \rangle$, $\langle \langle y', \langle i', j' \rangle \rangle, w', h', s', k' \rangle$ or $\langle \langle \langle i', j' \rangle, \langle i'', j'' \rangle \rangle, w', h', s', k' \rangle$ in dom (f^p) such that

$$f^{p}(\langle\langle i',j'\rangle,y'\rangle,w',h',s',k') = \langle i,j_{1}\rangle \text{ or } f^{p}(\langle y',\langle i',j'\rangle\rangle,w',h',s',k') = \langle i,j_{1}\rangle$$

holds for some $j_1 \leq j$, then

$$= \left(\left(\bigcup_{j \le j} z_{\langle i,j \rangle} \cup \bigcup_{j \le j'} z_{\langle i',j \rangle} \right) \cap \omega^{\le h^q} \right) \cup \left\{ s_{\langle i,j \rangle}; j \le j \right\} \cup \left\{ s_{\langle i',j \rangle}; j \le j' \right\},$$

• if there is $\langle\langle\langle i', j'\rangle, \langle i'', j''\rangle\rangle, w', h', s', k'\rangle$ in dom (f^p) such that $f^p(\langle\langle i', j'\rangle, \langle i'', j''\rangle\rangle, w', h', s', k') = \langle i, j_1 \rangle$ holds for some $j_1 \leq j$, then

$$x^q_{\langle i,j\rangle}(l^p)\cap\omega^{\leq h^q}$$

$$= \left(\left(\bigcup_{j \le j} z_{\langle i,j \rangle} \cup \bigcup_{j \le j'} z_{\langle i',j \rangle} \cup \bigcup_{j \le j''} z_{\langle i'',j \rangle} \right) \cap \omega^{\le h^q} \right) \\ \cup \left\{ s_{\langle i,j \rangle}; j \le j \right\} \cup \left\{ s_{\langle i',j \rangle}; j \le j' \right\} \cup \left\{ s_{\langle i'',j \rangle}; j \le j'' \right\}$$

• otherwise,

$$x^{q}_{\langle i,j\rangle}(l^{p})\cap\omega^{\leq h^{q}} = \left(\left(\bigcup_{j\leq j} z_{\langle i,j\rangle}\right)\cap\omega^{\leq h^{q}}\right)\cup\left\{s_{\langle i,j\rangle};j\leq j\right\}.$$

We let

$$\langle x_{\sigma}^{q} \upharpoonright l^{p}; \sigma \in m^{q} \times n^{q} \rangle = \langle x_{\sigma}^{p}; \sigma \in (m^{p} + 1) \times n^{p} \rangle$$
$$w^{q} := w^{p} \cup \bigcup_{\sigma \in (m^{p} + 1) \times n^{p}} \left(z_{\sigma} \cap \omega^{h^{q}} \right),$$

and $f^q = f^p$. Then by the above observation, q is a condition of $\mathbb{P}(L_{<\alpha})$. So q is a desired extension of p.

Proposition 18. (Dense sets which guarantee that each $\boldsymbol{x}_{\langle i,j\rangle}^{G}$ is defined)

- (1) For each $n \in \omega$, the set $\{p \in \mathbb{P}(L_{<\alpha}); n^p \ge n\}$ is dense in $\mathbb{P}(L_{<\alpha})$.
- (2) For each $l \in \omega$, the set $\{p \in \mathbb{P}(L_{<\alpha}); l^p \ge l\}$ is dense in $\mathbb{P}(L_{<\alpha})$.
- (3) For each $h \in \omega$, the set $\{p \in \mathbb{P}(L_{<\alpha}); h^p \ge h\}$ is dense in $\mathbb{P}(L_{<\alpha})$.
- (4) For each $m \in \omega$, the set $\{p \in \mathbb{P}(L_{<\alpha}); m^p \ge m\}$ is dense in $\mathbb{P}(L_{<\alpha})$.

Proof. (1) For a given $p \in \mathbb{P}(L_{<\alpha})$, if $n^p < n$, then for each $i \in m^p$ and $j \in [n^p, n)$, we define $x_{\langle i,j \rangle} := x_{\langle i,n^p-1 \rangle}^p$. Then the tuple

$$\langle \{\langle \sigma, x^p_\sigma \rangle; \sigma \in m^p \times n^p\} \cup \{\langle \tau, x_\tau \rangle; \tau \in m^p \times [n^p, n)\}, w^p, h^p, f^p \rangle$$

is an extension of p.

(2) For a given $p \in \mathbb{P}(L_{<\alpha})$, if $l^p < l$, then for each $\sigma \in m^p \times n^p$ and $k \in l$, we define

$$x_{\sigma}(k) := \begin{cases} x_{\sigma}^{p}(k) & \text{if } k < l^{p} \\ x_{\sigma}^{p}(l^{p} - 1) & \text{if } k \ge l^{p} \end{cases}.$$

Then the tuple

$$\langle \langle x_{\sigma}; \sigma \in m^p \times n^p \rangle w^p, h^p, f^p \rangle$$

is an extension of p.

(3) For a given $p \in \mathbb{P}(L_{\leq \alpha})$, if $h^p < h$, then the tuple

$$\langle \langle x^p_{\sigma}; \sigma \in m^p \times n^p \rangle w^p, h, f^p \rangle$$

is an extension of p.

(4) This is a corollary of Lemma 17.

Proposition 19. (Dense sets which guarantee that each $\boldsymbol{x}_{\langle i,j \rangle}^G$ has rank $\alpha + j$ in $L_{\langle \alpha+\omega}$)

(1) For each $\gamma < \alpha$ and $\sigma \in \omega \times \omega$, the set

$$\left\{ p \in \mathbb{P}(L_{<\alpha}); \sigma \in m^p \times n^p \text{ and } \operatorname{ht}(x^p_{\sigma}(l^p - 1)) \ge \gamma \right\}$$

is dense in $\mathbb{P}(L_{\leq \alpha})$.

(2) For each i and j in ω , the set

$$\left\{ p \in \mathbb{P}(L_{<\alpha}); \langle i,j \rangle \in m^p \times n^p \text{ and } \left(x^p_{\langle i,j+1 \rangle}(l^p-1) \setminus x^p_{\langle i,j \rangle}(l^p-1) \right) \cap \omega^{\leq h^p} \neq \emptyset \right\}$$

is dense in $\mathbb{P}(L_{<\alpha})$.

Proof. These are corollaries of Lemma 17, because we have the property $A(L_{<\alpha})$.

Proposition 20. (Dense sets which guarantee that $L_{<\alpha+\omega}$ is closed under intersections)

(1) For each $z \in L_{<\alpha}$ and $\sigma \in \omega \times \omega$, the set

$$\left\{ p \in \mathbb{P}(L_{<\alpha}); \sigma \in m^p \times n^p \text{ and } \langle x^p_\sigma(l^p - 1) \cap z, w^p, h^p \rangle \notin S \right\}$$

is dense in $\mathbb{P}(L_{<\alpha})$.

(2) For each σ and τ in $\omega \times \omega$, the set

$$\left\{ p \in \mathbb{P}(L_{<\alpha}); \sigma \in m^p \times n^p, \ \tau \in m^p \times n^p, \ and \right\}$$

either " p decides that $\boldsymbol{x}_{\sigma}^{G}$ and $\boldsymbol{x}_{\tau}^{G}$ are comparable "

or " p decides that \pmb{x}_{σ}^{G} and \pmb{x}_{τ}^{G} are not comparable and

$$\left(x_{\sigma}^{p}(l^{p}-1)\setminus x_{\tau}^{p}(l^{p}-1)\right)\cap\omega^{\leq h^{p}}\neq\emptyset"\right\}$$

is dense in $\mathbb{P}(L_{<\alpha})$.

Proof. Let $p \in \mathbb{P}(L_{<\alpha})$. By extending p if necessary, we may assume that $\sigma \in m^p \times n^p$. By applying Lemma 17 to p and the set $(x^p_{\sigma}(l^p - 1) \cap z) \cap \omega^{h^p}$, we find a desired extension.

Proposition 21. (Dense sets which guarantee that $L_{<\alpha+\omega}$ still has the property A)

(1) For each $y \in L_{<\alpha}$, $\sigma \in \omega \times \omega$, $h \in \omega$, $w \in [\omega^{<\omega}]^{<\aleph_0}$ with $w \subseteq \omega^{\leq h}$, $s \in \omega^{<\omega}$ with |s| > h, and $j \in \omega$ $j \ge (2nd \ cor. \ of \ \sigma) + 1$, the set

$$\begin{split} \left\{ p \in \mathbb{P}(L_{<\alpha}); \sigma \in m^p \times n^p, w^p \supseteq w, h^p \ge h, \text{ and} \\ & \quad (x^p_{\sigma}(l^p - 1), w, h) \not \in \mathcal{S}^+ \\ & \text{or} \quad s \restriction h \notin x^p_{\sigma}(l^p - 1) \setminus \bigcup_{t \in w} \text{cone}(t) \\ & \text{or} \quad y \restriction_{w,h} \not \subseteq x^p_{\sigma}(l^p - 1) \restriction_{w,h} \\ & \text{or} \quad \langle \langle y, \sigma \rangle, w, h, s, j \rangle \in \text{dom}(f^p) \text{ "} \right\} \end{split}$$

is dense in $\mathbb{P}(L_{<\alpha})$.

(2) For each $\sigma \in \omega \times \omega$, $\langle y, w, h \rangle \in \left(L_{<\alpha} \times [\omega^{<\omega}]^{<\aleph_0} \omega \right) \cap S^+$, $s \in \omega^{<\omega}$ with |s| > h and $s \upharpoonright h \notin y \setminus \bigcup_{t \in w} \operatorname{cone}(t)$, and $j \in \omega$ $j \ge (2nd \ cor. \ of \ \sigma) + 1$, the set

$$\begin{split} \left\{ p \in \mathbb{P}(L_{<\alpha}); \sigma \in m^p \times n^p, w^p \supseteq w, h^p \ge h, \ and \\ & \quad `` \quad x^p_{\sigma}(l^p - 1) \restriction_{w,h} \not\subseteq y \restriction_{w,h} \\ & \quad \text{or} \quad \left\langle \left\langle \sigma, y \right\rangle, w, h, s, j \right\rangle \in \operatorname{dom}(f^p) " \quad \right\} \end{split}$$

is dense in $\mathbb{P}(L_{<\alpha})$.

(3) For each $\sigma, \tau \in \omega \times \omega$, $h \in \omega$, $w \in [\omega^{<\omega}]^{<\aleph_0}$ with $w \subseteq \omega^{\leq h}$, $s \in \omega^{<\omega}$ with |s| > h, and $j \in \omega$ $j \ge (2nd \ cor. \ of \ \sigma) + 1$, the set

$$\begin{cases} p \in \mathbb{P}(L_{<\alpha}); \sigma, \tau \in m^p \times n^p, w^p \supseteq w, h^p \ge h, \text{ and} \\ & \quad \langle x^p_\tau(l^p - 1), w, h \rangle \not\in \mathcal{S}^+ \\ & \text{or} \quad s \upharpoonright h \notin x^p_\tau(l^p - 1) \setminus \bigcup_{t \in w} \text{cone}(t) \\ & \text{or} \quad x^p_\sigma(l^p - 1) \upharpoonright_{w,h} \not\subseteq x^p_\tau(l^p - 1) \upharpoonright_{w,h} \\ & \text{or} \quad \langle \langle \sigma, \tau \rangle, w, h, s, j \rangle \in \text{dom}(f^p) \text{ "} \end{cases} \end{cases}$$

is dense in $\mathbb{P}(L_{<\alpha})$.

Proof. These proof are almost same to the proof of Lemma 17. The only difference is that if possible, we put on a tuple in the domain of a function f^p . \dashv

Proposition 22. (Dense sets for powerfully c.c.c.) Assume that \diamondsuit holds, and let \mathcal{X} be a \diamondsuit -sequence. If $n \in \omega$ and A is \trianglelefteq -mc in $L_{<\alpha}{}^n$, then for each $\overline{\sigma} \in (\omega \times \omega)^n$,

 $w \in [\omega^{<\omega}]^{<\aleph_0}$ and $h \in \omega$, the set

$$\begin{cases} p \in \mathbb{P}(L_{<\alpha}); \operatorname{ran}(\overline{\sigma}) \subseteq m^p \times n^p, w^p \supseteq w, h^p \ge h, \text{ and} \\ \text{``either} \quad \langle x^p_{\sigma}(l^p - 1)(k), w, h \rangle \not\in \mathcal{S} \text{ for some } k \in n \\ \text{or} \quad \exists \overline{a} \in A \text{ such that } \langle \overline{a}, w, h \rangle \in \mathcal{S}^n \text{ and} \\ \text{ for any } k \in n, \ \langle \overline{a}(k), w, h \rangle \trianglelefteq \langle x^p_{\overline{\sigma}(k)}(l^p - 1), w, h \rangle \text{ ''} \end{cases}$$

is dense in $\mathbb{P}(L_{<\alpha})$.

Proof. This is a corollary of Lemma 17 and the property $A(L_{<\alpha})$. Even if $\sigma(k) =$ $\sigma(k')$ holds for some different k and k' in n, we can build an extension because of the property $A(L_{\leq \alpha})$. (This point is quite different from the construction of a Suslin tree from \Diamond .)

Proof that $L_{<\omega_1}$ is a powerfully c.c.c. well-founded lower semi-lattice. Suppose that $n \in \omega$ and A is an uncountable \leq^n -mc in $L_{\leq \omega_1}^n$. Then there exists $\alpha \in \omega_1$ such that

- $A \cap L_{<\alpha}^n$ is \leq^n -mc in $L_{<\alpha}^n$, and
- ρ_α(A ∩ L_{<α}ⁿ) = ρ_{ω1}(A) ∩ L_{<α}ⁿ,
 ρ_α(A ∩ L_{<α}ⁿ) is coded by the α-th coordinate of the ◊-sequence X.

Then $\rho_{\alpha}(A \cap L_{<\alpha}{}^n) \in M_{\beta}$ for every $\beta \in [\alpha, \omega_1)$ (because both \mathcal{X} and α are members of M_{β} for every $\beta \in [\alpha, \omega_1)$). So by the density arguments (Proposition 22), $A \cap L_{<\alpha}{}^n$ is $\leq n$ -mc in $L_{<\omega_1}{}^n$, and $A \cap L_{<\alpha}{}^n = \rho_{\omega_1}(A \cap L_{<\alpha}{}^n) = \rho_{\alpha}(A \cap L_{<\alpha}{}^n)$, which is countable. This is a contradiction.

6. A model where there is a Suslin lower semi-lattice and all ARONSZAJN TREES ARE SPECIAL

In this section, we give a negative answer to Question 4. The strategy is to start with L as constructed in either Section 4 or 5 and to specialize all Aronszajn trees while preserving L. The idea is that L is so different from a Suslin tree that this can be done. We do a finite support (FS) iteration of c.c.c. forcings, and at each stage we consider an Aronszajn tree T and attempt to specialize T by a c.c.c. forcing. If this specializing forcing kills L, then we show that there is a c.c.c. forcing which adds a cofinal branch to T while preserving L, so that T is no longer an Aronszajn tree. It is here that the fact that L is powerfully c.c.c. is used.

The following two lemmas are folklore. We include the proof for the reader.

Lemma 23. Suppose (\mathbb{P}, \leq) is a powerfully c.c.c. poset. Fix $n \in \omega$. Let $\mathbb{Q} \subset \mathbb{P}^n$. Then $\Vdash_{\mathbb{O}}\mathbb{P}$ is powerfully c.c.c..

Proof. Fix $m \in \omega$. Let \mathring{X} be a \mathbb{Q} name such that $\Vdash \mathring{X} \subset \mathbb{P}^m$. Suppose $q \in \mathbb{Q}$ and $q \Vdash X$ is an uncountable p.i.e.. Find $\{r_{\alpha} : \alpha < \omega_1\} \subset \mathbb{Q}$ and $\{\sigma_{\alpha} : \alpha < \omega_1\} \subset \mathbb{P}^m$ such that for each $\alpha < \omega_1$, $r_{\alpha} \leq q$, $\sigma_{\alpha} \notin \{\sigma_{\xi} : \xi < \alpha\}$, and $r_{\alpha} \Vdash \sigma_{\alpha} \in \mathring{X}$. For each $\alpha < \omega_1$, put $\tau_{\alpha} = \sigma_{\alpha} \frown r_{\alpha} \in \mathbb{P}^{m+n}$. As \mathbb{P} is powerfully c.c.c. there exist $\xi, \alpha \in \omega_1$ such that $\xi \neq \alpha$ and $\tau_{\alpha} \leq \tau_{\xi}$. But then $r_{\alpha} \leq r_{\xi}$ and so $r_{\alpha} \Vdash \sigma_{\alpha} \in X \land \sigma_{\xi} \in X$. On the other hand, $\sigma_{\alpha} \leq \sigma_{\xi}$, and since $\alpha \neq \xi$, $\sigma_{\alpha} \neq \sigma_{\xi}$, contradicting the hypothesis that $q \Vdash X$ is a p.i.e.. \dashv **Lemma 24.** Suppose (L, \triangleleft) is powerfully c.c.c. Suppose $(\mathbb{P}_{\alpha}, \mathbb{Q}_{\alpha} : \alpha \leq \delta)$ is a FS iteration of c.c.c. forcing notions such that for each $\alpha < \delta$

 $\Vdash_{\alpha} \Vdash_{\mathring{\mathbb{O}}_{\alpha}} \langle L, \triangleleft \rangle \text{ is powerfully c.c.c.}$

Then $\Vdash_{\delta} \langle L, \triangleleft \rangle$ is powerfully c.c.c.

Proof. The proof is by induction on δ . When δ is 0 or a successor, there is nothing to prove. So assume δ is a limit and that the statement holds for all smaller ordinals. Fix $n \in \omega$. Let $\mathring{X} \in \mathbf{V}^{\mathbb{P}_{\delta}}$ such that $\Vdash_{\delta} \mathring{X} \subset L^{n}$. Fix $p \in \mathbb{P}_{\delta}$, and suppose $p \Vdash_{\delta} X$ is an uncountable p.i.e.. Find $\{p_{\alpha} : \alpha < \omega_1\}$ and $\{\sigma_{\alpha} : \alpha < \omega_1\} \subset L^n$ such that for each $\alpha < \omega_1, \ p_\alpha \leq p, \ \sigma_\alpha \notin \{\sigma_{\xi} : \xi < \alpha\}, \ p_\alpha \Vdash_{\delta} \sigma_\alpha \in \mathring{X}$. Without loss of generality, there is $F \in [\delta]^{<\omega}$ such that $\{\operatorname{suppt}(p_{\alpha}) : \alpha < \omega_1\}$ forms a Δ -system with root F. Fix $\gamma < \delta$ such that $F \subset \gamma$. Note that for any $\xi < \alpha < \omega_1$, if σ_{ξ} and σ_{α} are comparable under \triangleleft , then $p_{\xi} \perp_{\delta} p_{\alpha}$, and so $p_{\xi} \upharpoonright \gamma \perp_{\gamma} p_{\alpha} \upharpoonright \gamma$. Now, for each $\alpha < \omega_1, p_{\alpha} \upharpoonright \gamma \leq p \upharpoonright \gamma$. So by a standard argument, there is $q \in \mathbb{P}_{\gamma}$ such that $q \leq p \upharpoonright \gamma$ and $q \Vdash_{\gamma} \{ \alpha < \omega_1 : p_\alpha \upharpoonright \gamma \in \mathring{G}_{\gamma} \}$ is uncountable. Take a $(\mathbf{V}, \mathbb{P}_{\gamma})$ -generic filter G_{γ} with $q \in G_{\gamma}$. Put $Y = \{ \alpha < \omega_1 : p_{\alpha} \upharpoonright \gamma \in G_{\gamma} \}$. For any $\xi < \alpha$ in $Y, p_{\xi} \upharpoonright \gamma$ and $p_{\alpha} \upharpoonright \gamma$ are compatible, and so σ_{ξ} and σ_{α} are incomparable under \triangleleft . But then $\{\sigma_{\alpha} : \alpha \in Y\} \subset L^n$ is an uncountable p.i.e. in $\mathbf{V}[G_{\gamma}]$, contradicting the inductive hypothesis.

Lemma 25. Let T be an ω_1 -Aronszajn tree. Then there can be no sequence $\langle F_{\alpha} :$ $\alpha < \omega_1$ and $m \in \omega$ such that

- (1) $F_{\alpha} \in [T]^{<\omega}$
- (2) for all $\alpha < \beta < \omega_1 \forall x \in F_{\alpha} \forall y \in F_{\beta} [\operatorname{ht}(x) < \operatorname{ht}(y)]$ (3) for any $H \in [\omega_1]^m$, there exist $\alpha, \beta \in H$ such that $\alpha < \beta$ and $\exists x \in F_{\alpha} \exists y \in F_{\alpha}$. $F_{\beta} [x < y].$

Proof. Note that for (3) to hold, m must be at least 2. When m = 2, the result follows by a well known argument of Baumgartner, Malitz, and Reinhardt [3]. So assume that m > 2 and that the result holds for m - 1. Fix $\langle F_{\alpha} : \alpha <$ $|\omega_1\rangle$ satisfying (1)-(3). It follows that for any $\delta < \omega_1$, $\exists H \in [\omega_1 \setminus \delta]^{m-1} \forall \alpha, \beta \in$ H [if $\alpha < \beta$, then $\forall x \in F_{\alpha} \forall y \in F_{\beta} [x \not< y]$]. Now, reduce it to the case m = 2 as follows. Select $\langle H_{\xi} : \xi < \omega_1 \rangle$ and $\langle G_{\xi} : \xi < \omega_1 \rangle$ such that

- (4) $H_{\xi} \in [\omega_1]^{m-1}$ and $G_{\xi} = \bigcup_{\alpha \in H_{\xi}} F_{\alpha}$ (5) $\forall \xi < \zeta < \omega_1 \forall \alpha \in H_{\xi} \forall \beta \in H_{\zeta} [\alpha < \beta]$ (6) $\forall \xi < \omega_1 \forall \alpha, \beta \in H_{\xi} [\text{if } \alpha < \beta, \text{ then } \forall x \in F_{\alpha} \forall y \in F_{\beta} [x \not< y]]$
- (7) $\forall \xi < \zeta < \omega_1 \exists x \in G_{\xi} \exists y \in G_{\zeta} [x < y].$

To see that this is possible, suppose $\langle H_{\eta} : \eta < \xi \rangle$ and $\langle G_{\eta} : \eta < \xi \rangle$ are given. Put $\delta = \sup \left(\bigcup_{\eta < \xi} H_{\eta} \right)$. Choose $H_{\xi} \in [\omega_1 \setminus \delta + 1]^{m-1}$ such that (6) holds and put $G_{\xi} = \bigcup_{\alpha \in H_{\xi}} \check{F}_{\alpha}$. To see that (7) holds, fix $\eta < \xi$. Then $|H_{\eta} \cup H_{\xi}| \ge m$. Because of (6), there must be $\alpha \in H_{\eta}$, $\beta \in H_{\xi}$, $x \in F_{\alpha}$, and $y \in F_{\beta}$ such that x < y. \dashv

Let T be an Aronszajn tree. Recall the usual poset for specializing T, which we will denote \mathbb{P}_T . A condition p in \mathbb{P}_T is a function $p: F_p \to \mathbb{Q}$, where $F_p \in$ $[T]^{<\omega}, \mathbb{Q}$ is the set of rational numbers, and p has the property that $\forall x, y \in \mathbb{R}$ $F_p[x < y \implies p(x) < p(y)]$. And $q \leq p$ iff $q \supset p$. It is well known that \mathbb{P}_T is c.c.c. and that it adds a specializing map from T to \mathbb{Q} .

Lemma 26. Let $L \subset \mathcal{P}(\omega)$ be such that $\langle L, \subsetneq \rangle$ is a Suslin lower semi-lattice of height ω_1 . Assume moreover that $\langle L, \subset \rangle$ is powerfully c.c.c. Let $\langle T, \triangleleft \rangle$ be an Aronszajn tree. Suppose

 $\not\Vdash_{\mathbb{P}_{\mathcal{T}}} \langle L, \subset \rangle \text{ is powerfully c.c.c.}$

Then there is $n \in \omega$ and $M \subset L^n$ such that

(1) $\Vdash_{\langle M, \supset \rangle} T$ has a cofinal branch

(2) $\Vdash_{\langle M, \supset \rangle} \langle L, \subset \rangle$ is powerfully c.c.c.

Proof. Fix $n \in \omega$. Let \mathring{Z} be a \mathbb{P}_T name such that $\Vdash \mathring{Z} \subset L^n$, and let $p \in \mathbb{P}_T$ be so that $p \Vdash \mathring{Z}$ is an uncountable p.i.e.. Choose $\{p_\alpha : \alpha < \omega_1\}$ and $\{\sigma_\alpha : \alpha < \omega_1\}$ such that for each $\alpha < \omega_1$, $p_\alpha \leq p$, $\sigma_\alpha \notin \{\sigma_\xi : \xi < \alpha\}$, and $p_\alpha \Vdash \sigma_\alpha \in \mathring{Z}$. For each $\alpha < \omega_1$, write F_α for F_{p_α} . Find $X \in [\omega_1]^{\omega_1}$, $R \in [T]^{<\omega}$, $l \in \omega$, and $f : R \to \mathbb{Q}$ such that

- (3) $\{F_{\alpha} : \alpha \in X\}$ forms a Δ -system with root R
- (4) for each $\alpha, \beta \in X$, if $\alpha < \beta$, then for all $s \in F_{\alpha} \setminus R$, $t \in F_{\beta} \setminus R$, and $u \in R$, ht(u) < ht(s) < ht(t)
- (5) $\forall \alpha \in X [p_{\alpha} \upharpoonright R = f \land |F_{\alpha} \setminus R| = l].$

Now, fix $\xi < \alpha$ in X. If σ_{ξ} and σ_{α} are comparable under \subset , then p_{ξ} and p_{α} must be incompatible in \mathbb{P}_T , and hence, there must be $s \in F_{\xi} \setminus R$ and $t \in F_{\alpha} \setminus R$ such that $s \triangleleft t$. Now, $\{\alpha \in X : |\{\xi \in X : \sigma_{\xi} \text{ and } \sigma_{\alpha} \text{ are comparable under } \subset\}| \leq \omega\}$ must be countable because $\langle L^n, \subset \rangle$ does not have an uncountable p.i.e. Let Y be $\{\alpha \in X : \{\xi \in X : \sigma_{\xi} \text{ and } \sigma_{\alpha} \text{ are comparable under } \subset\}$ is uncountable}. Put $M = \{\sigma_{\alpha} : \alpha \in Y\}$. Consider $\langle M, \supset \rangle$ as a forcing notion. It has the c.c.c. Also, for each $\alpha \in Y$, there are uncountably many $\xi \in Y$ such that σ_{α} and σ_{ξ} are compatible conditions. So by a standard argument $\Vdash_M \{ \alpha \in Y : \sigma_\alpha \in G_M \}$ is uncountable, where G_M is the canonical *M*-name for a (\mathbf{V}, M) -generic filter. Now, we claim that $\Vdash_M T$ has a cofinal branch. Fix a (\mathbf{V}, M) -generic filter G, and work inside $\mathbf{V}[G].\ Y^*=\{\alpha\in Y:\sigma_\alpha\in G\}$ is uncountable. We check that (1)-(3) of Lemma 25 are satisfied by $\langle F_{\alpha} \setminus R : \alpha \in Y^* \rangle$. To check (3) of Lemma 25, fix $H \in [Y^*]^{l+1}$. There is $\xi \in Y^*$ such that $\forall \alpha \in H [\sigma_\alpha \subset \sigma_\xi]$. Now, there are uncountably many $\zeta \in X$ such that σ_{ζ} and σ_{ξ} are comparable under \subset . On the other hand, $\{\tau \in L^n : \tau \subset \sigma_{\xi}\}$ is countable. So it is possible to find $\zeta \in X$ such that $\forall \alpha \in H \ [\alpha < \zeta \land \sigma_{\alpha} \subset \sigma_{\zeta}]$. So for each $\alpha \in H$, there is $s_{\alpha} \in F_{\alpha} \setminus R$ and $t_{\alpha} \in F_{\zeta} \setminus R$ such that $s_{\alpha} \triangleleft t_{\alpha}$. Since $|F_{\zeta} \setminus R| = l$, and |H| = l + 1, it follows that there are $\alpha, \beta \in H$ with $\alpha < \beta$ such that $t_{\alpha} = t_{\beta}$. But then, $s_{\alpha} \triangleleft s_{\beta}$, as needed.

For (2), note that for any poset $\langle \mathbb{P}, \leq \rangle$, $\langle \mathbb{P}, \leq \rangle$ is powerfully c.c.c. iff $\langle \mathbb{P}, \geq \rangle$ is also powerfully c.c.c. So since $M \subset L^n$, and $\langle L, \supset \rangle$ is powerfully c.c.c., it follows from Lemma 23 that $\Vdash_{\langle M, \supset \rangle} \langle L, \supset \rangle$ is powerfully c.c.c., whence (2). \dashv

Theorem 27. It is consistent that there is a Suslin lower semi-lattice and yet all Aronszajn trees are special.

Proof. Start with a ground model \mathbf{V} where there is $L \subset \mathcal{P}(\omega)$ such that $\langle L, \subsetneq \rangle$ is a Suslin lower semi-lattice such that $\operatorname{ht}(L) = \omega_1, \forall x, y \in L [x \land y = x \cap y]$, and $\langle L, \subset \rangle$ is powerfully c.c.c. Assume moreover that GCH holds in \mathbf{V} . Such a \mathbf{V} exists by the results of the previous sections. Using a suitable bookkeeping device to ensure that all names for Aronszajn trees are eventually considered, do a FS iteration $\langle \mathbb{P}_{\alpha}, \mathbb{Q}_{\alpha} : \alpha \leq \omega_2 \rangle$ of c.c.c. posets as follows. At a stage $\alpha < \omega_2$, let \mathring{T} be a \mathbb{P}_{α}

name for an Aronszajn tree given by the bookkeeping device. Fix a $(\mathbf{V}, \mathbb{P}_{\alpha})$ -generic filter G, and work inside $\mathbf{V}[G]$. If $\Vdash_{\mathbb{P}_{\hat{T}[G]}} \langle L, \subset \rangle$ is powerfully c.c.c., then let \mathbb{Q} be $\mathbb{P}_{\hat{T}[G]}$. Otherwise, use Lemma 26 to find $n \in \omega$ and $M \subset L^n$ such that (1) and (2) of Lemma 26 hold. Now, put $\mathbb{Q} = \langle M, \supset \rangle$. In either case, back in \mathbf{V} , let $\hat{\mathbb{Q}}_{\alpha}$ be a \mathbb{P}_{α} name for \mathbb{Q} . In the end, $\Vdash_{\omega_2} \langle L, \subset \rangle$ is powerfully c.c.c. So $\langle L, \subsetneq \rangle$ has no uncountable p.i.e. On the other hand, it can never have an uncountable chain. So $\langle L, \subsetneq \rangle$ is Suslin.

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