Q-POINTS IN THE TUKEY ORDER

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ABSTRACT. Q-points are cofinal in the RK-ordering under several mild hypotheses.

1. INTRODUCTION

The purpose of this note is to address a question that has been considered in two recent papers, namely the existence of Tukey maximal Q-points. This was asked in [2] and addressed in [3], where it was proved by a forcing argument that Tukey maximal Q-points may consistently exist. We show here that fairly standard constructions using mild hypotheses yield many Tukey maximal Q-points. If $\mathfrak{d} = \aleph_1$ or if there are infinitely many pairwise RK-non-isomorphic selective ultrafilters, then the Q-points are cofinal in the RK ordering. In particular, there are $2^{\mathfrak{c}}$ pairwise RK-incomparable Tukey maximal Q-points under either of these assumptions.

For general facts about the Tukey theory of directed sets we refer the reader to [19] or [18]. For the specific case of ultrafilters on ω, see, for example, [14] or [5]. The order structure of P-points in the RK and Tukey ordering is considered in [12], [8], [15], [9], among other places. Consistency results can be found in [17], [16]. Our notation is standard. We refer to [10] for any undefined terms.

2. Q-POINTS ARE COFINAL

Definition 2.1. An *interval partition or IP* is a sequence $I = \langle i_n : n \in \omega \rangle \in \omega^\omega$ such that $i_0 = 0$ and $\forall n \in \omega\, [i_n < i_{n+1}]$.

Given an IP I and $n \in \omega$, I_n denotes $[i_n, i_{n+1}) = \{l \in \omega : i_n \leq l < i_{n+1}\}$.

Recall that an ultrafilter $\mathcal{U}$ on ω is a *Q-point* if for every finite-to-one function $f : \omega \to \omega$, there exists $A \in \mathcal{U}$ such that f is one-to-one on A. It is well known (see e.g. [1]) that an ultrafilter $\mathcal{U}$ on ω is a Q-point if and only if for every IP I, there exists $A \in \mathcal{U}$ such that $\forall k \in \omega\, [|I_k \cap A| \leq 1]$.

Recall that $\mathfrak{d}$ is the minimal cardinality of a dominating family of functions in ω^ω. It is well-known (see e.g. [4]) that $\mathfrak{d}$ is the minimal κ such that there exists a sequence $\langle I_\alpha : \alpha < \kappa \rangle$ of IPs such that for every IP I, there exists $\alpha < \kappa$ such that $\forall l \in \omega \exists k \in \omega\, [I_k \subseteq I^\alpha_l]$. This is the characterization of $\mathfrak{d}$ which is useful below. Recall that Canjar [6] showed that if $\mathfrak{d} = \aleph_1$, then any filter on ω that is generated by fewer than $2^{\aleph_0}$ sets can be extended to a Q-point.

Lemma 2.2. *Suppose $\langle Y_n : n \in \omega \rangle$ is a sequence such that:*

(1) $Y_n \in [\omega]^{\aleph_0}$, for all $n \in \omega$;
(2) $Y_n \cap Y_m = \emptyset$, for all $n < m < \omega$;

Suppose $I = \langle i_n : n \in \omega \rangle$ is an IP. Then there exists $X \in [\omega]^{\aleph_0}$ such that:

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(3) $|X \cap Y_n| = \aleph_0$, *for all* $n \in \omega$;
(4) $|X \cap I_l| \leq 1$, *for all* $l \in \omega$.

Proof. Define $Z_n = \{l \in \omega : I_l \cap Y_n \neq \emptyset\}$, for all $n \in \omega$. $Z_n \in [\omega]^{\aleph_0}$ because $Y_n \in [\omega]^{\aleph_0}$ and because I is an IP. Let $\langle T_n : n \in \omega \rangle$ be such that:

(5) $T_n \in [Z_n]^{\aleph_0}$, for all $n \in \omega$;
(6) $T_m \cap T_n = \emptyset$, for all $m < n < \omega$.

For each $l \in T_n$, since $I_l \cap Y_n \neq \emptyset$, choose $k_{l,n} \in I_l \cap Y_n$. Let

$$X = \{k_{l,n} : l \in T_n \wedge n \in \omega\}.$$

For each $n \in \omega$, $\{k_{l,n} : l \in T_n\} \subseteq X \cap Y_n$, and if $l \neq l'$, then $k_{l,n} \neq k_{l',n}$ because $k_{l,n} \in I_l$, $k_{l',n} \in I_{l'}$, and $I_l \cap I_{l'} = \emptyset$. Therefore, (3) is satisfied. For (4), suppose that $n, n' \in \omega$, $l \in T_n$, $l' \in T_{n'}$, and that $k_{l,n}, k_{l',n'} \in I_j$, for some $j \in \omega$. Then since $k_{l,n} \in I_j \cap I_l$ and $k_{l',n'} \in I_j \cap I_{l'}$, $l = j = l'$. Since $l \in T_n \cap T_{n'}$, $n = n'$, whence $k_{l,n} = k_{l',n'}$. Hence (4) is satisfied and X is as needed. $\dashv$

Lemma 2.3. *Suppose* $\langle I^\alpha : \alpha < \omega_1 \rangle$ *is a sequence of IPs. Suppose also that* $\langle X_n : n \in \omega \rangle$ *is a sequence such that:*

(1) $X_n \in [\omega]^{\aleph_0}$, *for all* $n \in \omega$;
(2) $X_m \cap X_n = \emptyset$, *for all* $m < n < \omega$;
(3) $\omega = \bigcup_{n\in\omega} X_n$.

There exists a filter $\mathcal{F}$ *on* ω *such that:*

(4) $\forall Z \in \mathcal{F} \forall n \in \omega \left[|Z \cap X_n| = \aleph_0\right]$;
(5) $\forall \alpha < \omega_1 \exists Z \in \mathcal{F} \forall l \in \omega \left[|I^\alpha_l \cap Z| \leq 1\right]$.

Proof. For any $Z \subseteq \omega$, let $Z[n]$ denote $Z \cap X_n$. Build by induction a family $\{Z_\alpha : \alpha < \omega_1\} \subseteq [\omega]^{\aleph_0}$ satisfying the following:

(6) $\forall \alpha < \omega_1 \forall n \in \omega \left[|Z_\alpha[n]| = \aleph_0\right]$;
(7) $\forall \alpha < \beta < \omega_1 \forall n \in \omega \left[Z_\beta[n] \subseteq^* Z_\alpha[n]\right]$;
(8) $\forall \alpha < \omega_1 \forall l \in \omega \left[|I^\alpha_l \cap Z_\alpha| \leq 1\right]$.

Suppose for a moment that this has been accomplished. Let

$$\mathcal{F} = \left\{Z \subseteq \omega : \exists \alpha < \omega_1 \forall n \in \omega \left[Z_\alpha[n] \subseteq^* Z[n]\right]\right\}.$$

Then it is clear that $\mathcal{F}$ is a filter on ω. If $Z \in \mathcal{F}$ as witnessed by $\alpha < \omega_1$, then for each $n \in \omega$, $Z_\alpha[n] \subseteq^* Z[n]$, so by (6), (4) holds. Since for each $\alpha < \omega_1$, $Z_\alpha \in \mathcal{F}$, (8) gives (5). Hence $\mathcal{F}$ is a required.

To build $\{Z_\alpha : \alpha < \omega_1\}$, proceed by induction. Fix $\alpha < \omega_1$ and suppose $\{Z_\xi : \xi < \alpha\} \subseteq [\omega]^{\aleph_0}$ satisfying (6)–(8) is given. For each $n \in \omega$, the family $\{Z_\xi[n] : \xi < \alpha\} \subseteq [X_n]^{\aleph_0}$ satisfies $\forall \zeta < \xi < \alpha \left[Z_\xi[n] \subseteq^* Z_\zeta[n]\right]$. As α is countable, find $Y_n \in [X_n]^{\aleph_0}$ such that $\forall \xi < \alpha \left[Y_n \subseteq^* Z_\xi[n]\right]$. Note that $Y_m \cap Y_n = \emptyset$ for $n \neq m$ because $X_m \cap X_n = \emptyset$. Applying Lemma 2.2, find $Z \in [\omega]^{\aleph_0}$ satisfying (3) and (4) of Lemma 2.2 for I^α. Define $Z_\alpha = Z \cap \left(\bigcup_{m\in\omega} Y_m\right)$. It is clear that (8) follows from (4) of Lemma 2.2. Note that for any $n \in \omega$, $Z_\alpha[n] = Z \cap Y_n$. Therefore, (6) follows from (3) of Lemma 2.2 and (7) follows from the choice of Y_n. Thus Z_α has all the required properties. This concludes the construction and the proof. $\dashv$

Theorem 2.4. *Assume* $\mathfrak{d} = \aleph_1$. *Let* $\mathcal{U}$ *be any ultrafilter on* ω *and let* $f : \omega \to \omega$ *be such that* $\left|f^{-1}(\{n\})\right| = \aleph_0$, *for all* $n \in \omega$. *There there exists* $\mathcal{V}$ *such that:*

(1) $\mathcal{V}$ *is a Q-point;*
(2) f *witnesses that* $\mathcal{U} \leq_{RK} \mathcal{V}$.

Proof. Fix a sequence $\langle I^\alpha : \alpha < \omega_1 \rangle$ of IPs so that for every IP I, there exists $\alpha < \omega_1$ such that $\forall l \in \omega \exists k \in \omega\, [I_k \subseteq I^\alpha_l]$. Define $X_n = f^{-1}(\{n\})$, for every $n \in \omega$. Applying Lemma 2.3, find a filter $\mathcal{F}$ on ω satisfying (4) and (5) of Lemma 2.3. Suppose $Z \in \mathcal{F}$ and $A \in \mathcal{U}$. Then $f^{-1}(A) = \bigcup_{n \in A} X_n$, whence $Z \cap f^{-1}(A) = \bigcup_{n \in A}(Z \cap X_n)$, which is infinite by (4) of Lemma 2.3. It follows that there exists an ultrafilter $\mathcal{V}$ on ω such that $\mathcal{F} \subseteq \mathcal{V}$ and $\{f^{-1}(A) : A \in \mathcal{U}\} \subseteq \mathcal{V}$. Now (2) is immediate by the choice of $\mathcal{V}$. To see (1), let I be any IP. Let $\alpha < \omega_1$ be such that $\forall l \in \omega \exists k \in \omega\, [I_k \subseteq I^\alpha_l]$. By (5) of Lemma 2.3, there exists $Z \in \mathcal{F} \subseteq \mathcal{V}$ such that $\forall l \in \omega\, [|I^\alpha_l \cap Z| \leq 1]$. It is easily seen that $\forall n \in \omega\, [|I_n \cap Z| \leq 2]$. Define $R = \{\min(I_n \cap Z) : n \in \omega \wedge I_n \cap Z \neq \emptyset\}$ and $S = Z \setminus R$. It is clear that $\forall n \in \omega\, [|I_n \cap R| \leq 1]$ and that $\forall n \in \omega\, [|I_n \cap S| \leq 1]$. $R \in \mathcal{V}$ or $S \in \mathcal{V}$ because $Z \in \mathcal{V}$ and $\mathcal{V}$ is an ultrafilter. Therefore, $\mathcal{V}$ is a Q-point (see [13, Lemma 7.1] for a similar argument). $\dashv$

Corollary 2.5. *Assume $\mathfrak{d} = \aleph_1$. There exist $\left\langle \mathcal{V}_\alpha : \alpha < 2^{2^{\aleph_0}} \right\rangle$ such that:*

(1) $\mathcal{V}_\alpha$ is a Q-point for every $\alpha < 2^{2^{\aleph_0}}$;
(2) $\mathcal{V}_\alpha \not\leq_{RK} \mathcal{V}_\beta$, for $\alpha, \beta < 2^{2^{\aleph_0}}$ with $\alpha \neq \beta$;
(3) $\mathcal{V}_\alpha$ is Tukey maximal, for all $\alpha < 2^{2^{\aleph_0}}$.

Proof. There exists in ZFC a family of ultrafilters $\{\mathcal{U}_\alpha : \alpha < 2^{2^{\aleph_0}}\}$ on ω such that each $\mathcal{U}_\alpha$ is Tukey maximal and $\mathcal{U}_\alpha \neq \mathcal{U}_\beta$, for $\alpha \neq \beta$. Fix $f : \omega \to \omega$ such that $\left|f^{-1}(\{n\})\right| = \aleph_0$. By Theorem 2.4, find $\mathcal{V}_\alpha$ satisfying (1) and (2) of Theorem 2.4 for f and $\mathcal{U}_\alpha$. Each $\mathcal{V}_\alpha$ is a Q-point which is Tukey maximal as it is RK-above $\mathcal{U}_\alpha$. Since $\mathcal{U}_\alpha = \{A \subseteq \omega : f^{-1}(A) \in \mathcal{V}_\alpha\}$, it follows that $\mathcal{V}_\alpha \neq \mathcal{V}_\beta$, whenever $\alpha \neq \beta$. For each $\alpha < 2^{2^{\aleph_0}}$, let $F(\alpha) = \{\beta < 2^{2^{\aleph_0}} : \mathcal{V}_\beta \leq_{RK} \mathcal{V}_\alpha\}$. Since $|F(\alpha)| \leq 2^{\aleph_0}$, by a fundamental theorem on set mappings (see e.g. [7]), there exists $X \subseteq 2^{2^{\aleph_0}}$ with $|X| = 2^{2^{\aleph_0}}$ such that for each $\alpha \in X$, $F(\alpha) \cap X = \{\alpha\}$. Therefore, $\langle \mathcal{V}_\alpha : \alpha \in X \rangle$ is as needed. $\dashv$

It is easy to see that the construction in Theorem 2.4 can also be carried out if there are infinitely many pairwise RK-non-isomorphic selective ultrafilters. It is well-known that this happens if, for example, $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. We refer the reader to [11] for a discussion of the differences between constructions from hypotheses of the form $\mathfrak{d} = \aleph_1$ and those from hypotheses of the form $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. We give a few details below.

Lemma 2.6. *Suppose $\mathcal{U}$ and $\mathcal{V}$ are selective ultrafilters on ω with $\mathcal{U} \not\equiv_{RK} \mathcal{V}$. For any IP I, there exist $X \in \mathcal{U}$ and $Y \in \mathcal{V}$ such that:*
(1) $\forall n \in \omega\, [|I_n \cap X| \leq 1]$ and $\forall n \in \omega\, [|I_n \cap Y| \leq 1]$;
(2) $\{n \in \omega : I_n \cap X \neq \emptyset\} \cap \{n \in \omega : I_n \cap Y \neq \emptyset\} = \emptyset$.

Proof. Find $A \in \mathcal{U}$ and $B \in \mathcal{V}$ such that $\forall n \in \omega\, [|I_n \cap A| \leq 1]$ and

$$\forall n \in \omega\, [|I_n \cap B| \leq 1].$$

Let $f : \omega \to \omega$ be the function such that $f''I_n = \{n\}$, for all $n \in \omega$. Let $\mathcal{U}^* = \{U \subseteq \omega : f^{-1}(U) \in \mathcal{U}\}$ and let $\mathcal{V}^* = \{V \subseteq \omega : f^{-1}(V) \in \mathcal{V}\}$. Then $\mathcal{U}^* \neq \mathcal{V}^*$ because $\mathcal{U}^* \equiv_{RK} \mathcal{U} \not\equiv_{RK} \mathcal{V} \equiv_{RK} \mathcal{V}^*$. So there exist $U \in \mathcal{U}^*$ and $V \in \mathcal{V}^*$ with $U \cap V = \emptyset$. Then $X = A \cap f^{-1}(U) \in \mathcal{U}$ and $Y = B \cap f^{-1}(V) \in \mathcal{V}$ are as required. $\dashv$

Corollary 2.7. *Assume there exists a family $\{\mathcal{U}_n : n \in \omega\}$ such that:*
(1) $\mathcal{U}_n$ is a selective ultrafilter on ω;
(2) for each $n, m \in \omega$, if $n \neq m$, then $\mathcal{U}_m \not\equiv_{RK} \mathcal{U}_n$.
Let $\mathcal{U}$ be any ultrafilter on ω and let $f : \omega \to \omega$ be such that $\left|f^{-1}(\{n\})\right| = \aleph_0$, for all $n \in \omega$. There there exists $\mathcal{V}$ such that:

(3) $\mathcal{V}$ *is a Q-point;*
(4) f *witnesses that* $\mathcal{U} \leq_{RK} \mathcal{V}$.

Proof. Define $X_n = f^{-1}(\{n\})$. Let $\mathcal{V}_n$ be an RK-isomorphic copy of $\mathcal{U}_n$ with $X_n \in \mathcal{V}_n$, and define

$$\mathcal{V} = \{A \subseteq \omega : \{n \in \omega : A \cap X_n \in \mathcal{V}_n\} \in \mathcal{U}\}.$$

$\mathcal{V}$ is as required. To see this, suppose I is any IP. Using Lemma 2.6, fix $A_{m,n} \in \mathcal{V}_m$ and $B_{m,n} \in \mathcal{V}_n$ satisfying (1) and (2) of Lemma 2.6, for all $m < n < \omega$. For each $m \in \omega$, find $C_m \in \mathcal{V}_m$ with $C_m \subseteq A_{m,m+1}$ and $\forall m < n < \omega\, [C_m \subseteq^* A_{m,n}]$. Note $\forall k \in \omega\, [|I_k \cap C_m| \leq 1]$. For each $n \in \omega$, define $D_n = C_n \cap \left(\bigcap_{m<n} B_{m,n}\right) \in \mathcal{V}_n$, with $\bigcap \emptyset$ taken to be ω. For each $n \in \omega$, find $k_n \in \omega$ such that $\forall m < n \left[D_m \setminus A_{m,n} \subseteq \bigcup_{l \leq k_n} I_l\right]$. Define $E_n = D_n \setminus \left(\bigcup_{l \leq k_n} I_l\right) \in \mathcal{V}_n$ and define $E = \bigcup_{n \in \omega} (E_n \cap X_n)$. It is easily seen that $E \in \mathcal{V}$ and that $\forall l \in \omega\, [|I_l \cap E| \leq 1]$. Therefore, (3) is satisfied and it is clear from the definition of $\mathcal{V}$ that (4) holds as well (see [10] for a similar argument). ⊣

Corollary 2.8. *If there are infinitely many pairwise RK-non-isomorphic selective ultrafilters, then there are* $2^{\mathfrak{c}}$ *pairwise RK-incomparable Tukey maximal Q-points.*

It is well-known (see e.g. [6]) that if $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$, then every filter base on ω of size $< \mathfrak{c}$ can be extended to a selective ultrafilter, in particular, the hypotheses of Corollary 2.7 hold when $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. In [3] it is proved that a Tukey maximal Q-point exists when κ Cohen reals are added to a ground model satisfying CH. Since $\mathrm{cov}(\mathcal{M}) = \kappa = \mathfrak{c}$ in such a model, Corollary 2.7 improves this result from [3].

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