The Journal of Symbolic Logic
Volume 00, Number 0, XXX 0000

GREGORY TREES, THE CONTINUUM, AND MARTIN'S AXIOM

KENNETH KUNEN AND DILIP RAGHAVAN

Abstract. We continue the investigation of Gregory trees and the Cantor Tree Property carried out by Hart and Kunen. We produce models of MA with the Continuum arbitrarily large in which there are Gregory trees, and in which there are no Gregory trees.

§1. Introduction. We view the tree $2^{<\omega_1}$ as a forcing poset, defining $p \leq q$ iff $p \supseteq q$; so $\mathbb{1} = \emptyset$, the empty sequence. A *Gregory tree* is a type of subtree of $2^{<\omega_1}$ which is "almost countably closed". The notion is due to Gregory [1], although the terminology in the next definition is from Hart and Kunen [3].

Definition 1.1. *A* Cantor tree *in* $2^{<\omega_1}$ *is a subset* $\{f_\sigma : \sigma \in 2^{<\omega}\} \subseteq 2^{<\omega_1}$ *such that for all* $\sigma \in 2^{<\omega}$, $f_{\sigma^\frown 0}$ *and* $f_{\sigma^\frown 1}$ *are incompatible nodes in* $2^{<\omega_1}$ *that extend* f_σ. *A subtree* $\mathbb{T}$ *of* $2^{<\omega_1}$ *has the* Cantor Tree Property (CTP) *iff*

1. *For every* $f \in \mathbb{T}$, $f^\frown 0, f^\frown 1 \in \mathbb{T}$.
2. *Given any Cantor tree* $\{f_\sigma : \sigma \in 2^{<\omega}\} \subset \mathbb{T}$, *there are* $x \in 2^\omega$ *and* $g \in \mathbb{T}$ *such that* $\forall n \in \omega\, [g \leq f_{x\restriction n}]$.

A subtree $\mathbb{T}$ *of* $2^{<\omega_1}$ *is a* Gregory tree *iff it has the CTP, but does not have a cofinal branch.*

Paper [4] relates Gregory trees to more general forcing posets with the CTP.

Theorem 1.2 (Gregory [1]). $2^{\aleph_0} < 2^{\aleph_1}$ *implies that there is a Gregory tree.*

Gregory trees are of interest in the theory of proper forcing. It is easy to see (Lemma 5.5 of [3]) that a Gregory tree $\mathbb{T}$ is a totally proper poset, that is, it is proper and does not add any reals. Moreover, forcing with $\mathbb{T}$ adds a cofinal branch through $\mathbb{T}$. One might hope to do a countable support iteration of these totally proper forcings, producing a model of CH plus no Gregory trees, but this is impossible by Theorem 1.2, so that the iteration must add reals, although the CTP is annoyingly close to being countably closed. Of course,

Proposition 1.3. PFA *implies that there are no Gregory trees.*

Whenever a result is proved from PFA, two natural questions arise. First, does it follow just from MA + $\neg$CH? Second, is it consistent with $2^{\aleph_0} > \aleph_2$? Of course, the second question is trivial if the answer to the first question is "yes". In this paper, with regard

Received by the editors August 26, 2008.
2000 *Mathematics Subject Classification.* 03E35.
Key words and phrases. Martin's Axiom, Gregory tree, proper forcing.
Authors partially supported by NSF Grant DMS-0456653.

0022-4812/00/0000-0000/$00.00

to Proposition 1.3, we show that the answer is "no" to the first question and "yes" to the second. In Section 3, we produce models of MA in which there exists a Gregory tree; $\mathfrak{c}$ can be "anything regular". In Section, 4 we produce models of MA + $\neg$CH in which there does not exist a Gregory tree; here, $\mathfrak{c}$ can be "anything regular" *except* the successor of a cardinal of cofinality ω, so we are left with the following:

QUESTION 1.4. *Assume* MA *and* $\mathfrak{c} = \aleph_{\omega+1}$. *Must there be a Gregory tree?*

Gregory trees have arisen naturally in two different ways in topology.

First, by Lemma 5.7 of [3], if X is compact, hereditarily Lindelöf, and not totally disconnected, and X has no subspaces homeomorphic to the Cantor set, then there is a Gregory tree; the tree is contained in the forcing poset of non-trivial closed connected subsets of X. We do not know if the existence of a Gregory tree implies that there is such a space X, but there is such an X under $\diamondsuit$ by Theorem 1.3 of [3].

Second, a Gregory tree exists *if and only if* there is a compact totally disconnected Hausdorff space X with the CTP which is the union of $\aleph_1$ nowhere dense sets. Here, we say that X has the CTP iff the forcing poset of clopen subsets of X ordered by inclusion has the CTP in the sense described in [4]. The $\leftarrow$ direction is clear from [4]. To prove the $\rightarrow$ direction, let X be the set of maximal chains in a Gregory tree ordered lexicographically; then X is a compact LOTS with the CTP; the α^{th} nowhere dense set is the set of chains of height α. The CTP for X is a weakening of the stronger property that the forcing poset of clopen subsets of X is countably closed (non-empty G_δ subsets of X have non-empty interiors), which holds of many familiar spaces, such as $\beta\mathbb{N}\backslash\mathbb{N}$. A compactum with this stronger property is never the union of $\aleph_1$ nowhere dense sets.

§2. Notational Conventions for Iterated c.c.c. Forcing. In this paper we only consider finite support iterations of c.c.c. forcings. Before giving the proofs of our theorems, we set out some notational conventions regarding these iterations.

As usual in forcing, a *forcing poset* $\mathbb{P}$ really denotes a triple, $(\mathbb{P}, \leq, \mathbb{1})$, where $\leq$ is a transitive reflexive relation on $\mathbb{P}$ and $\mathbb{1}$ is a largest element of $\mathbb{P}$. Then, the notation $\mathbb{P} \subseteq \mathbb{Q}$ implies that the orders agree and that $\mathbb{1}_{\mathbb{P}} = \mathbb{1}_{\mathbb{Q}}$. $\mathbb{P} \subseteq_c \mathbb{Q}$ means that in addition, $\mathbb{P}$ is a *complete* sub-order of $\mathbb{Q}$; this implies that we may view the $\mathbb{Q}$–extension as a generic extension of the $\mathbb{P}$–extension (see, e.g., [5]). Since all our iterated forcings are c.c.c. with finite supports, it is simpler *not* to follow the approach of [5], but rather to construct in the ground model a *normal chain* of c.c.c. posets, $\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$, where $\alpha < \beta \rightarrow \mathbb{P}_\alpha \subseteq_c \mathbb{P}_\beta$ and we take unions at limits (which preserves the c.c.c.). In standard iterated forcing constructions, the $\mathbb{P}_\alpha$ are constructed inductively; given $\mathbb{P}_\alpha$, we choose $\mathring{\mathbb{Q}}_\alpha$, which is a $\mathbb{P}_\alpha$–name forced by $\mathbb{1}$ to be a c.c.c. poset; then $\mathbb{P}_{\alpha+1}$ is identified with $\mathbb{P}_\alpha * \mathring{\mathbb{Q}}_\alpha$. However, the basic theory of these iterations does not require a $\mathring{\mathbb{Q}}_\alpha$; in Section 4, it will sometimes be convenient to view a γ–chain as a $\mathrm{cf}(\gamma)$–chain by restricting to a cofinal sequence.

We shall always take $\mathbb{P}_0 = \{\mathbb{1}\}$, so that we can identify the $\mathbb{P}_0$–extension with the ground model. If G is a $(\mathbf{V}, \mathbb{P}_\kappa)$–generic filter, then $G_\alpha := G \cap \mathbb{P}_\alpha$ is $(\mathbf{V}, \mathbb{P}_\alpha)$–generic, and we let $\mathbf{V}_\alpha = \mathbf{V}[G_\alpha]$; so, $\mathbf{V}_0 = \mathbf{V}$.

If φ is a sentence in the $\mathbb{P}_\alpha$ forcing language and $p \in \mathbb{P}_\alpha$, then $p \Vdash_\alpha \varphi$ abbreviates $p \Vdash_{\mathbb{P}_\alpha} \varphi$. Note that we need a subscript on the $\Vdash$, since for any $\beta > \alpha$, the assertion "$p \Vdash_\beta \varphi$" is meaningful, although its truth can vary with β.

We use Shoenfield–style names as in [5]; that is, a name is a set of ordered pairs of names and forcing conditions. So, an inclusion of names $(\mathring{A} \subseteq \mathring{B})$ implies an inclusion of the sets named $(\mathbb{1} \Vdash \mathring{A} \subseteq \mathring{B})$. Also, if $\mathbb{P}_\alpha \subseteq_c \mathbb{P}_\beta$, then every $\mathbb{P}_\alpha$–name is a $\mathbb{P}_\beta$–name. In Section 3, we

shall build a Gregory tree $\mathbb{T}$ in $\mathbf{V}[G]$ by constructing in $\mathbf{V}$ an ascending sequence of names $\langle \mathring{\mathbb{T}}_\alpha : \alpha \leq \kappa \rangle$, where $\mathring{\mathbb{T}}_\alpha$ is a $\mathbb{P}_\alpha$–name.

If G is $\mathbb{P}$–generic over $\mathbf{V}$ and $X \in \mathbf{V}$, then every subset of X in $\mathbf{V}[G]$ is named by a *nice name $\mathring{b}$ for a subset of X*; so $\mathring{b} = \bigcup\{\{\check{x}\} \times E_x : x \in X\}$, where each E_x is an antichain in $\mathbb{P}$ (see [5]). Also, if $p \Vdash \mathring{a} \subseteq \check{X}$ then there is a nice name $\mathring{b}$ for a subset of X such that $p \Vdash \mathring{a} = \mathring{b}$. With iterated forcing, where $\mathbb{P} = \mathbb{P}_\gamma$ results from a normal chain of c.c.c. posets $\langle \mathbb{P}_\alpha : \alpha \leq \gamma \rangle$: if $\mathrm{cf}(\gamma) \geq \omega_1$ and X is countable, then, since the antichains are also countable, there is an $\alpha_0 < \gamma$ such that our $\mathring{b}$ is also a nice $\mathbb{P}_\alpha$ name whenever $\alpha_0 \leq \alpha \leq \gamma$.

§3. A Model of MA + ¬CH + There is a Gregory Tree.

THEOREM 3.1. *Assume that in the ground model $\mathbf{V}$, $\kappa \geq \aleph_2$ and $\kappa^{<\kappa} = \kappa$. Then there is a c.c.c. forcing extension $\mathbf{V}[G]$ satisfying $\mathrm{MA} + 2^{\aleph_0} = \kappa$ in which there is a Gregory tree.*

PROOF. The standard procedure for constructing a model of MA in which some consequence of PFA fails is to start with a counter-example in $\mathbf{V}$ which is not destroyed by the c.c.c. iteration. However, every Gregory tree $\mathbb{T}$ in $\mathbf{V}$ is destroyed immediately whenever a real is added, since that will cause the CTP to fail. Instead, our tree $\mathbb{T}$ will grow along with the iterated forcing which produces our model. To do this, we inductively construct the following, satisfying the listed conditions:

1. $\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$ is a normal chain of c.c.c. posets.
2. $|\mathbb{P}_\alpha| < \kappa$ for all $\alpha < \kappa$.
3. $\mathbb{P}_{\alpha+1} \cong \mathbb{P}_\alpha * \mathrm{Fn}(\omega, 2) * \mathring{\mathbb{Q}}_\alpha$, where $\mathbb{1} \Vdash_\alpha$ "$\mathring{\mathbb{Q}}_\alpha$ is c.c.c.".
4. Each $\mathring{\mathbb{T}}_\alpha$ is a $\mathbb{P}_\alpha$–name, for $\alpha \leq \kappa$.
5. $\mathbb{1} \Vdash_\alpha$ "$\mathring{\mathbb{T}}_\alpha$ is a subtree of $2^{<\omega_1}$ and $\forall f \in \mathring{\mathbb{T}}_\alpha \forall s \in 2^{<\omega}\, [f^\frown s \in \mathring{\mathbb{T}}_\alpha]$".
6. $\mathring{\mathbb{T}}_0$ is a name for the Cantor tree $2^{<\omega}$.
7. If $\alpha < \beta$ then $\mathring{\mathbb{T}}_\alpha \subseteq \mathring{\mathbb{T}}_\beta$, so that $\mathbb{1} \Vdash_\beta \mathring{\mathbb{T}}_\alpha \subseteq \mathring{\mathbb{T}}_\beta$.
8. If γ is a limit, then $\mathring{\mathbb{T}}_\gamma = \bigcup_{\alpha<\gamma} \mathring{\mathbb{T}}_\gamma$.
9. $\mathring{g}_\alpha$ is a $\mathbb{P}_{\alpha+1}$–name and $\mathbb{1} \Vdash_{\alpha+1}$ "$\mathring{\mathbb{T}}_{\alpha+1} = \mathring{\mathbb{T}}_\alpha \cup \{\mathring{g}_\alpha^\frown s : s \in 2^{<\omega}\}$"
10. $\mathbb{1} \Vdash_\alpha$ "$\mathring{\mathbb{T}}_\alpha$ has no uncountable chains".
11. $\mathbb{1} \Vdash_{\alpha+1}$ "$\mathring{\mathbb{T}}_\alpha$ is special".
12. $\mathring{c}_\alpha$ is a $\mathbb{P}_{\alpha+1}$–name for the function from ω to 2 added by the $\mathrm{Fn}(\omega, 2)$ in item (3).
13. $\mathring{K}_\alpha$ and $\mathring{k}_{\alpha,s}$ are $\mathbb{P}_\alpha$–names whenever $\alpha < \omega_1$ and $s \in 2^{<\omega}$.
14. $\mathbb{1} \Vdash_\alpha$ "$\mathring{K}_\alpha$ is a Cantor tree in $\mathring{\mathbb{T}}_\alpha$, and $\mathring{K}_\alpha$ is indexed in the standard way as $\{\mathring{k}_{\alpha,s} : s \in 2^{<\omega}\}$".
15. $\mathbb{1} \Vdash_{\alpha+1}$ "$\mathring{g}_\alpha = \bigcup\{k_{\alpha,\mathring{c}_\alpha \restriction n} : n \in \omega\}$".

Ignoring the "$\mathrm{Fn}(\omega, 2)$", Conditions (1)(2)(3) are the standard setup for forcing MA. We apply the usual bookkeeping to make sure that the $\mathring{\mathbb{Q}}_\alpha$ run through names for all possible c.c.c. orders of size $< \kappa$; then $\mathbf{V}_\kappa$ satisfies $\mathrm{MA} + 2^{\aleph_0} = \kappa$. This all still works if we include the "$\mathrm{Fn}(\omega, 2)$", which we use to construct the Gregory tree.

Conditions $(1 - 10)$ give us the Gregory tree $\mathbb{T}$ in $\mathbf{V}_\kappa$, named by $\mathring{\mathbb{T}}_\kappa$. Condition (10) implies that $\mathbb{T}$ has no uncountable chains, and the usual bookkeeping would let us choose the $\mathring{g}_\alpha$ so that every Cantor subtree of $\mathbb{T}$ has a path. The main difficulty in the construction is in preserving (10). There are problems both at successors and at limits, addressed by Conditions $(11 - 15)$.

At successors: Since $\mathring{\mathbb{T}}_{\alpha+1}$ is forced to be a subtree of $2^{<\omega_1}$, (9) requires $\mathring{g}_\alpha$ to be forced to be in $2^{<\omega_1}$, with all proper initial segments of $\mathring{g}_\alpha$ in $\mathring{\mathbb{T}}_\alpha$. Since $\mathring{\mathbb{T}}_{\alpha+1} \backslash \mathring{\mathbb{T}}_\alpha$ is forced to be

countable, Condition (10) is preserved in passing from $\mathring{\mathbb{T}}_\alpha$ to $\mathring{\mathbb{T}}_{\alpha+1}$ *unless* $\mathring{\mathbb{Q}}_\alpha$ adds a path through $\mathring{\mathbb{T}}_\alpha$, but this cannot happen by (11).

There is no problem ensuring (11) in the inductive construction. $\mathrm{Fn}(\omega, 2)$ can never add a path through $\mathbb{T}_\alpha$. To make sure that $\mathring{\mathbb{Q}}_\alpha$ does not add such a path, let $\mathring{\mathbb{Q}}_\alpha \cong \mathring{\mathbb{S}}_\alpha * \mathring{\mathbb{R}}_\alpha$, where $\mathring{\mathbb{S}}_\alpha$ is the name for the poset which specializes $\mathring{\mathbb{T}}_\alpha$. Note that this does not interfere with the usual bookkeeping for making MA true. Say this bookkeeping tells us that $\mathring{\mathbb{Q}}_\alpha$ should be $\mathring{\mathbb{B}}_\alpha$, which we may assume is always a $\mathbb{P}_\alpha$–name and that $\mathbb{1} \Vdash_\alpha$ "$\mathring{\mathbb{B}}_\alpha$ is c.c.c."; the c.c.c. is not affected by the $\mathrm{Fn}(\omega, 2)$, but it could be affected by the specializing order. Then $\mathring{\mathbb{R}}_\alpha$ is a $\mathbb{P}_\alpha * \mathrm{Fn}(\omega, 2) * \mathring{\mathbb{S}}_\alpha$ name for the partial order which is $\mathring{\mathbb{B}}_\alpha$ if $\mathring{\mathbb{B}}_\alpha$ remains c.c.c. after forcing with $\mathring{\mathbb{S}}_\alpha$, and otherwise is the trivial order $\{\mathbb{1}\}$.

At limits: In (8), we are literally taking the union of names in the ground model. This clearly preserves (4)(5) for $\mathbb{T}_\gamma$, and (10) is also preserved *unless* $\mathrm{cf}(\gamma) = \omega_1$, in which case (10) might fail. For example, the g_α for $\alpha < \omega_1$ might all be compatible, yielding an uncountable chain in $\mathbb{T}_{\omega_1}$.

We avoid this problem by $(12 - 15)$. These say that working in $\mathbf{V}_{\alpha+1}$, we choose the node $g_\alpha \in \mathbb{T}_{\alpha+1}$ as follows: We take a Cantor tree $K_\alpha \subseteq \mathbb{T}_\alpha$ (given to us by the usual bookkeeping) and let g_α be the path through this Cantor tree indexed by the Cohen real c_α added into $\mathbf{V}_{\alpha+1}$ by the $\mathrm{Fn}(\omega, 2)$. Since $K_\alpha \in \mathbf{V}_\alpha$ by (13),

$$g_\alpha \notin \mathbf{V}_\alpha. \tag{a}$$

Now, suppose that (10) should fail at some point during the construction. Then we have $\gamma \leq \kappa$ such that (10) holds for all $\alpha < \gamma$ but (10) fails at γ, so that we have a $\mathbb{P}_\gamma$–name $\mathring{h}$ and a $p \in \mathbb{P}_\gamma$ which forces that $\mathring{h} \in 2^{\omega_1}$ and is a path through $\mathring{\mathbb{T}}_\gamma$; we may assume that $\mathring{h}$ is a nice name for a subset of $\omega_1 \times 2$. As noted above, γ is a limit of cofinality ω_1. Now, we argue both in $\mathbf{V}$ and in $\mathbf{V}[G]$, where $p \in G$ and G is $(\mathbf{V}, \mathbb{P}_\gamma)$–generic.

In $\mathbf{V}$, let $\langle \alpha_\xi : \xi < \omega_1 \rangle$ be a continuously increasing sequence of limit ordinals with supremum γ. For $\mu < \omega_1$, we regard $\mathring{h}{\restriction}\mu$ as a nice $\mathbb{P}_\gamma$–name for an element of 2^μ; since $\mathbb{P}_\gamma$ is c.c.c., this $\mathring{h}{\restriction}\mu$ is actually a $\mathbb{P}_{\alpha_\xi}$–name for some $\xi < \omega_1$. Then there is a club $C_0 \subset \omega_1$ such that $\mathring{h}{\restriction}\alpha_\xi$ is a $\mathbb{P}_{\alpha_\xi}$–name for each $\xi \in C_0$; so, in $\mathbf{V}[G]$, we have

$$h{\restriction}\alpha_\xi \in \mathbf{V}_{\alpha_\xi}. \tag{b}$$

Also in $\mathbf{V}[G]$, each $\mathbb{T}_{\alpha_\xi}$ is special, so there is an $\eta > \xi$ such that $h{\restriction}\alpha_\eta \notin \mathbb{T}_{\alpha_\xi}$. Since we are taking unions of the trees at limit ordinals, there is a club $C_1 \subset \omega_1$ such that for $\xi \in C_1$ we have

$$h{\restriction}\alpha_\xi \notin \mathbb{T}_{\alpha_\xi}. \tag{c}$$

Fix a limit ordinal $\xi \in C_0 \cap C_1$. Since $h{\restriction}\alpha_\xi \in \mathbb{T}_\gamma$, we may fix δ with $\alpha_\xi \leq \delta < \gamma$ such that $h{\restriction}\alpha_\xi \in \mathbb{T}_{\delta+1} \backslash \mathbb{T}_\delta$, which implies, by (9), that $h{\restriction}\alpha_\xi = g_\delta$, so $g_\delta \in \mathbf{V}_\delta$ by (b), contradicting (a). ⊣

§4. Consistency of no Gregory Trees with Large Continuum. In this section we shall prove:

THEOREM 4.1. *Assume that in the ground model* $\mathbf{V}$*:*

1. $\kappa \geq \aleph_2$ *and* $\kappa^{<\kappa} = \kappa$.
2. $\lambda^{\aleph_0} < \kappa$ *for all* $\lambda < \kappa$.
3. $\diamondsuit_\kappa(S)$*, where* $S = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega_1\}$.

Then there is a c.c.c. forcing extension $\mathbf{V}[G]$ *satisfying* $\mathrm{MA} + 2^{\aleph_0} = \kappa$ *in which there are no Gregory trees.*

We do not know whether (3) follows from (1) and (2); it does by Gregory [2] in the case that $\kappa = \lambda^+$ and $\lambda^{\aleph_1} = \lambda$. If we start with $\mathbf{V} = \mathbf{L}$, then (1) and (3) hold for all regular $\kappa \geq \aleph_2$, but (2) fails if κ is the successor to a cardinal of cofinality ω, so we are left with Question 1.4.

As with the proof of Theorem 3.1, we shall modify the usual ccc iteration to produce a model of $\mathrm{MA}+2^{\aleph_0} = \kappa$ (using (1)). To kill a potential Gregory tree $\mathbb{T}$ in $\mathbf{V}[G]$, we use (2) plus countably closed elementary submodels to produce a club $C \subseteq \kappa$ such that $\mathbb{T}^\alpha := T \cap \mathbf{V}[G_\alpha]$ has the CTP in $\mathbf{V}[G_\alpha]$ for all $\alpha \in C \cap S$. Then, we use (3) to ensure that at some stage α in the construction, we kill $\mathbb{T}^\alpha$ by shooting a cofinal branch through it, so that we also kill $\mathbb{T}$.

Now, to kill $\mathbb{T}^\alpha$ by a c.c.c. poset, we cannot force with $\mathbb{T}^\alpha$, since this is not c.c.c. Instead, we shall find a Suslin subtree $\mathbb{Q}_\alpha \subset \mathbb{T}^\alpha$ and force with $\mathbb{Q}_\alpha$. This method is patterned after [3], which proved Theorem 4.1 in the special case that $\kappa = \aleph_2$ and $\diamondsuit$ (that is, $\diamondsuit_{\omega_1}$) holds in $\mathbf{V}$. It is well-known that $\diamondsuit$ will remain true in $\mathbf{V}[G_\alpha]$ (since $\alpha < \omega_2$), and hence, by Lemma 5.8 of [3], the tree $\mathbb{T}^\alpha$ will have a Suslin subtree. But for longer iterations, $\diamondsuit$ (and CH) will fail whenever $\alpha \geq \omega_2$. Instead, we shall use the fact that $\mathrm{cf}(\alpha) = \omega_1$. It is well-known that this implies that there is a Suslin tree in $\mathbf{V}[G_\alpha]$, since Cohen reals have been added cofinally often below α (see, for example, Theorems 3.1 and 6.1 of [6]). Here, we shall prove Theorem 4.7, which shows how to get the Suslin tree $\mathbb{Q}_\alpha$ inside of $\mathbb{T}^\alpha$.

Definition 4.2. *For a limit ordinal* γ *and a subtree* $\mathbb{T} \subseteq 2^{<\gamma}$*:* $\mathbb{T}$ *is* uniformly of height γ *iff* $\forall f \in \mathbb{T}\, \forall \alpha < \gamma\, \exists h \in \mathbb{T}\, [h < f \ \&\ \mathrm{ht}(h) > \alpha]$*, and* $\mathbb{T}$ *is* branchy *iff* $f^\frown 0, f^\frown 1 \in \mathbb{T}$ *for all* $f \in \mathbb{T}$*. If* $\mathbb{T} \in \mathbf{V}$ *and* $g : \gamma \to 2$*, then* g *is* $\mathbb{T}$–generic over $\mathbf{V}$ *iff* $\{g{\restriction}\alpha : \alpha < \gamma\}$ *is* $\mathbb{T}$*–generic over* $\mathbf{V}$*.*

Such a tree is an atomless forcing order, and every $\mathbb{T}$–generic filter is a path through $\mathbb{T}$. If $\mathbb{T}$ is countable, then $\mathbb{T}$ is equivalent to Cohen forcing $\mathrm{Fn}(\omega, 2)$. We can now modify the standard Jensen construction of a Suslin tree $\mathbb{T} \subseteq 2^{<\omega_1}$; the Cohen reals allow us to replace the use of $\diamondsuit$ at limits $\gamma < \omega_1$ by the requirement that all $g \in \mathbb{T} \cap 2^\gamma$ be $\mathbb{T} \cap 2^{<\gamma}$–generic. This is described in Lemma 4.4, which we shall prove after listing some further conventions for names in c.c.c. forcing extensions.

Say $\mathbb{P}$ is c.c.c. and $p \Vdash \mathring{a} \in 2^{<\omega_1}$. Then p may not decide what the height (= domain) $\mathrm{ht}(\mathring{a})$ is, but there is a $\xi < \omega_1$ such that $p \Vdash \mathrm{ht}(\mathring{a}) \leq \xi$, so $\mathring{a}$ is forced to be a subset of $\xi \times 2$, and there is a nice name $\mathring{b}$ for a subset of $\xi \times 2$ such that $p \Vdash \mathring{a} = \mathring{b}$.

Next, consider subsets $A \subseteq 2^{<\omega_1}$ in $\mathbf{V}[G]$; A may be a tree, or an antichain in a tree; again, $\mathbb{P}$ is c.c.c. A is not a subset of a ground model set, but we may simplify the name for A as follows. Say p forces that $\mathring{A} \subseteq 2^{<\omega_1}$ and $1 \leq |\mathring{A}| \leq \kappa$. Then, in $\mathbf{V}[G]$, we may list A in a κ–sequence (possibly with repetitions), so there is a name $\mathring{B}$ such that $p \Vdash \mathring{A} = \mathring{B}$, where $\mathring{B} = \{\langle \mathring{b}_\mu, p\rangle : \mu < \kappa\}$ and each $\mathring{b}_\mu$ is a nice name for a subset of some $\xi_\mu \times 2$, where $\xi_\mu < \omega_1$ and $\mathbb{1} \Vdash \mathring{b}_\mu \in 2^{<\omega_1} \wedge \mathrm{ht}(\mathring{b}_\mu) \leq \xi_\mu$.

These conventions make it easy to apply elementary submodel arguments in $\mathbf{V}$ to the forcing construction. For example,

Lemma 4.3. *Assume in* $\mathbf{V}$*:* $\kappa \geq \aleph_2$ *is regular and* $\lambda^{\aleph_0} < \kappa$ *for all* $\lambda < \kappa$ *and* $\langle \mathbb{P}_\alpha : \alpha \leq \kappa\rangle$ *is a normal chain of c.c.c. posets, with each* $|\mathbb{P}_\alpha| < \kappa$*. Let* $\mathring{\mathbb{T}} = \{\langle \mathring{b}_\mu, \mathbb{1}\rangle : \mu < \kappa\}$ *be a* $\mathbb{P}_\kappa$*–name which is forced by* $\mathbb{1}$ *to be a subtree of* $2^{<\omega_1}$ *with the CTP, where each* $\mathring{b}_\mu$ *is a nice name for a subset of some* $\xi_\mu \times 2$ *with* $\xi_\mu < \omega_1$*. Let* $\mathring{\mathbb{T}}^\alpha = \{\langle \mathring{b}_\mu, \mathbb{1}\rangle : \mu < \alpha\}$*.*

There is then a club $C \subseteq \kappa$ such that for all $\alpha \in C$ with $\mathrm{cf}(\alpha) > \omega$: $\mathring{\mathbb{T}}^\alpha$ is a $\mathbb{P}_\alpha$–name and $\mathbb{1} \Vdash_\alpha$ "$\mathring{\mathbb{T}}^\alpha$ is a subtree of $2^{<\omega_1}$ with the CTP".

Proof. Fix a suitably large regular θ. Given the assumptions on κ, it is sufficient to prove that the conclusion to the lemma holds whenever α is an ordinal of the form $M \cap \kappa$, where $M \prec H(\theta)$ is a countably closed elementary submodel containing the relevant objects.

The fact that $\mathring{\mathbb{T}}^\alpha$ is a $\mathbb{P}_\alpha$–name is immediate and does not need countable closure. Likewise, to show that $\mathbb{1} \Vdash_\alpha$ "$\mathring{\mathbb{T}}^\alpha$ is a subtree", note that for each $\mu < \kappa$ there is a countable $R \subseteq \kappa$ such that $\mathbb{1} \Vdash \forall \eta \leq \xi_\mu \exists \nu \in R\, [\mathring{b}_\mu \restriction \eta = \mathring{b}_\nu]$, and by $M \prec H(\theta)$, some such R is in M, so that $R \subset \alpha$. The proof of the CTP is similar, but uses the countable closure of M to imply that M contains $\mathbb{P}_\alpha$–names for every possible Cantor subtree of $\mathring{\mathbb{T}}^\alpha$ which lies in $\mathbf{V}_\alpha$. ⊣

Similar (and easier) reflection arguments work for sets A of size $\aleph_1$. Call $A \subseteq 2^{<\omega_1}$ *skinny* iff $|A| = \aleph_1$ and each $A \cap 2^\xi$ is countable. Then we can list A in an ω_1–sequence, listing nodes in order of their height. If p forces that $\mathring{A}$ is a skinny subset of ω_1, then there is a club C and a name $\mathring{B}$ such that $p \Vdash \mathring{A} = \mathring{B}$, where $\mathring{B} = \{\langle \mathring{b}_\mu, p\rangle : \mu < \omega_1\}$ as above, and also $\mathbb{1} \Vdash \mathrm{ht}(\mathring{b}_\mu) \geq \gamma$ whenever $\mu \geq \gamma \in C$. If $\mathring{B}_\gamma$ is the name $\{\langle \mathring{b}_\mu, p\rangle : \mu < \gamma\}$, then $\mathbb{1} \Vdash \mathring{B} \cap 2^{<\gamma} = \mathring{B}_\gamma$ whenever $\gamma \in C$. With iterated forcing, where $\mathbb{P} = \mathbb{P}_{\omega_1}$ results from a normal chain of c.c.c. posets $\langle \mathbb{P}_\alpha : \alpha \leq \omega_1\rangle$, we can also arrange for $\mathring{B}_\gamma$ to be a $\mathbb{P}_\gamma$–name whenever $\gamma \in C$, so that from the point of view of $\mathbf{V}[G]$ with $p \in G$, each $A \cap 2^{<\gamma} \in \mathbf{V}[G_\gamma]$.

Lemma 4.4. *Suppose that in $\mathbf{V}$, $\langle \mathbb{P}_\alpha : \alpha \leq \omega_1\rangle$ is a normal chain of c.c.c. posets and G is $\mathbb{P}_{\omega_1}$–generic. In $\mathbf{V}[G]$, suppose that the subtree $\mathbb{T} \subseteq 2^{<\omega_1}$ is uncountable, branchy, skinny, and uniformly of height ω_1. Assume also that there is club of limit ordinals $C \subset \omega_1$ such that for all $\gamma \in C$: $\mathbb{T} \cap 2^{<\gamma} \in \mathbf{V}[G_\gamma]$ and every $g \in \mathbb{T} \cap 2^\gamma$ is $\mathbb{T} \cap 2^{<\gamma}$–generic over $\mathbf{V}[G_\gamma]$. Then $\mathbb{T}$ is Suslin in $\mathbf{V}[G]$.*

Proof. If not, then in $\mathbf{V}[G]$ we have an uncountable maximal antichain $A \subseteq \mathbb{T}$. Then there is a club C_1 such that $A \cap 2^{<\gamma}$ is a maximal antichain in $\mathbb{T} \cap 2^{<\gamma}$ for all $\gamma \in C_1$. Also, A is skinny, so as pointed out above, there is a club C_2 such that $A \cap 2^{<\gamma} \in \mathbf{V}[G_\gamma]$ for all $\gamma \in C_2$. Now, fix $\gamma \in C \cap C_1 \cap C_2$, and consider any $g \in \mathbb{T} \cap 2^\gamma$. Then $\{g \restriction \xi : \xi < \gamma\}$ is generic over $\mathbf{V}[G_\gamma]$, so it meets the maximal antichain $A \cap 2^{<\gamma} \in \mathbf{V}[G_\gamma]$. But then $A \subseteq 2^{<\gamma}$, so A is countable. ⊣

This lemma lets us construct a Suslin tree in an iterated forcing extension, but we actually need our tree to be inside a given Gregory tree. To do that, we use the following lemma, which is related to the well-known fact that adding a Cohen real actually adds a perfect set of Cohen reals:

Lemma 4.5. *Assume that in $\mathbf{V}$: γ is a countable limit ordinal and $\mathbb{T}$ is a countable subtree of $2^{<\gamma}$ which is branchy and uniformly of height γ. Fix an ω–sequence $\langle \alpha_i : i < \omega\rangle$ of ordinals increasing to γ and fix any $f \in \mathbb{T}$ with $\mathrm{ht}(f) = \alpha_0$. Let $\mathbf{V}[G]$ be* any *forcing extension of $\mathbf{V}$ which contains a Cohen real (e.g., a filter which is $\mathrm{Fn}(\omega, 2)$–generic over $\mathbf{V}$).*

Then, in $\mathbf{V}[G]$: There is a Cantor tree $\{f_\sigma : \sigma \in 2^{<\omega}\} \subseteq \mathbb{T}$ such that $f_\emptyset = f$ and $\alpha_i \leq \mathrm{ht}(f_\sigma) < \gamma$ whenever $\mathrm{ht}(\sigma) = i$, and such that for all $\psi \in 2^\omega$, $\bigcup\{f_{\psi \restriction i} : i < \omega\}$ is $\mathbb{T}$–generic over $\mathbf{V}$.

Proof. In $\mathbf{V}$, let $\mathbb{P}$ be the poset of "partial Cantor trees starting at f". So, $p \in \mathbb{P}$ iff for some $n = n_p \in \omega$: p is a function from $2^{\leq n}$ into $\mathbb{T}$, $p(0) = f$, $\alpha_i \leq \mathrm{ht}(f_\sigma) < \gamma$ whenever $\mathrm{ht}(\sigma) = i \leq n$, and $p(\sigma^\frown 0) \perp p(\sigma^\frown 1)$ whenever $\mathrm{ht}(\sigma) < n$. Order $\mathbb{P}$ by $q \leq p$ iff $q \supseteq p$. $\mathbb{P}$ is countable, so the existence of a Cohen real implies the existence of a $(\mathbf{V}, \mathbb{P})$–generic

filter $H \in \mathbf{V}[G]$. Note that $\{p \in \mathbb{P} : n_p \geq m\}$ is dense for each m because $\mathbb{T}$ is uniformly of height γ, so $\bigcup H : 2^{<\omega} \to \mathbb{T}$. Let $f_\sigma = (\bigcup H)(\sigma)$. Fix any $\psi \in 2^\omega$. To verify $\mathbb{T}$–genericity of $\bigcup\{f_{\psi\restriction i} : i < \omega\}$, let $D \subseteq \mathbb{T}$ be dense. Then $D^* := \{p : \forall \sigma \in 2^{n_p}\, [p(\sigma) \in D]\}$ is dense in $\mathbb{P}$. If $p \in H \cap D^*$ then $f_{\psi\restriction n_p} \in D$. ⊣

In any non-trivial iterated forcing, the Cohen reals come for free because of the following well-known lemma.

LEMMA 4.6. *Suppose that in $\mathbf{V}$, γ is any limit ordinal and $\langle \mathbb{P}_\alpha : \alpha \leq \gamma\rangle$ is a normal chain of c.c.c. posets. Let G be $\mathbb{P}_\gamma$–generic over $\mathbf{V}$, and assume that $\mathbf{V}[G] \neq \mathbf{V}[G_\alpha]$ for any $\alpha < \gamma$. Then $\mathbf{V}[G]$ contains a real which is Cohen generic over $\mathbf{V}$.*

This lemma is actually not critical for our proof, since in iterating to make MA true, we could easily add a Cohen real explicitly at each stage.

THEOREM 4.7. *Suppose that in $\mathbf{V}$: π is a limit ordinal with $\mathrm{cf}(\pi) = \omega_1$, and $\langle \mathbb{P}_\alpha : \alpha \leq \pi\rangle$ is a normal chain of c.c.c. posets.*

Let G be $\mathbb{P}_\pi$–generic over $\mathbf{V}$, and assume that $\mathbf{V}[G] \neq \mathbf{V}[G_\alpha]$ for any $\alpha < \pi$.

In $\mathbf{V}[G]$: Let $\mathbb{T}$ be a subtree of $2^{<\omega_1}$ with the CTP. Then $\mathbb{T}$ has a Suslin subtree.

PROOF. First, restricting to a club and applying Lemma 4.6 (using the various $\mathbf{V}[G_\alpha]$ as the ground model), we may assume that $\pi = \omega_1$ and that each $\mathbf{V}[G_{\alpha+1}]$ contains a real which is Cohen generic over $\mathbf{V}[G_\alpha]$. Actually, the club is obtained in $\mathbf{V}[G]$; but it then contains a ground model club by the c.c.c.; so the restricted sequence of forcing posets also lies in $\mathbf{V}$.

Now, working in $\mathbf{V}[G]$, we construct a Suslin subtree $\mathbb{S} \subset \mathbb{T}$ by constructing inductively $\mathbb{S}\cap 2^{<\gamma}$. $\mathbb{S}$ will be branchy, so we only need to specify the construction for limit γ. We assume that we have $\mathbb{S}\cap 2^{<\gamma}$, and we assume (inductively) that each such $\mathbb{S}\cap 2^{<\gamma}$ is countable and is uniformly of height γ, and we must describe $\mathbb{S}\cap 2^\gamma$. For each $f \in \mathbb{S}\cap 2^{<\gamma}$, choose a $g_f \in 2^\gamma$ such that $g_f < f$ and such that $g_f\restriction\xi \in \mathbb{S}\cap 2^{<\gamma}$ for all $\xi < \gamma$; then $\mathbb{S}\cap 2^\gamma = \{g_f : f \in \mathbb{S}\cap 2^{<\gamma}\}$. To get each g_f: Fix an ω–sequence $\langle \alpha_i : i < \omega\rangle$ of ordinals increasing to γ, with $\alpha_0 = \mathrm{ht}(f)$. Then choose a Cantor tree $\{f_\sigma : \sigma \in 2^{<\omega}\} \subseteq \mathbb{S}\cap 2^{<\gamma}$ such that $f_\emptyset = f$ and $\alpha_i \leq \mathrm{ht}(f_\sigma) < \gamma$ whenever $\mathrm{ht}(\sigma) = i$; this is easily done since $\mathbb{S}$ is uniformly of height γ. Furthermore, *if* $\mathbb{S}\cap 2^{<\gamma} \in \mathbf{V}[G_\gamma]$, apply Lemma 4.5 and assume that for all $\psi \in 2^\omega$, $\bigcup\{f_{\psi\restriction i} : i < \omega\}$ is $(\mathbb{S}\cap 2^{<\gamma})$–generic over $\mathbf{V}[G_\gamma]$. Whether or not $\mathbb{S}\cap 2^{<\gamma} \in \mathbf{V}[G_\gamma]$, we apply the CTP of $\mathbb{T}$ to always choose $g_f = \bigcup_i f_{\psi\restriction i} \in \mathbb{T}$.

Now that we have constructed $\mathbb{S} \subset \mathbb{T}$ in $\mathbf{V}[G]$, we note that it is skinny, so as pointed out above, there is a club of limits $C \subset \omega_1$ such that $\mathbb{S}\cap 2^{<\gamma} \in \mathbf{V}[G_\gamma]$ for all $\gamma \in C$. For these γ, every $g \in \mathbb{S}\cap 2^\gamma$ is $\mathbb{S}\cap 2^{<\gamma}$–generic over $\mathbf{V}[G_\gamma]$, so, by Lemma 4.4, $\mathbb{S}$ is Suslin in $\mathbf{V}[G]$. ⊣

PROOF OF THEOREM 4.1. As with the proof of Theorem 3.1, in $\mathbf{V}$ we inductively construct the following, satisfying the listed conditions:

1. $\langle \mathbb{P}_\alpha : \alpha \leq \kappa\rangle$ is a normal chain of c.c.c. posets.
2. $|\mathbb{P}_\alpha| < \kappa$ for all $\alpha < \kappa$.
3. $\mathbb{P}_{\alpha+1} \cong \mathbb{P}_\alpha * \mathring{\mathbb{Q}}_\alpha$, where $\mathbb{1} \Vdash_\alpha$ "$\mathring{\mathbb{Q}}_\alpha$ is c.c.c.".
4. $\mathbb{P}_\alpha$ as a *set* is the ordinal $\delta_\alpha \leq \kappa$, with $\mathbb{1} = 0$.

As before, (1)(2)(3) are the standard setup for forcing MA. We apply the usual bookkeeping to make sure that the $\mathring{\mathbb{Q}}_\alpha$, for $\mathrm{cf}(\alpha) \neq \omega_1$, run through names for all possible atomless c.c.c. orders of size $< \kappa$, so $\mathbf{V}[G] = \mathbf{V}_\kappa$ satisfies MA + $2^{\aleph_0} = \kappa$. Condition (4) is irrelevant for this, although it is sometimes included in expositions to facilitate the bookkeeping. Note that κ is regular, so the δ_α, for $\alpha < \kappa$, form a continuously increasing sequences of ordinals

less than κ, and $\delta_\kappa = \kappa$; also note that $\{\alpha < \kappa : \delta_\alpha = \alpha\}$ is a club. We have included (4) to facilitate the use of $\diamondsuit_\kappa(S)$, which will give us $\mathbb{Q}_\alpha$ when $\mathrm{cf}(\alpha) = \omega_1$.

To show that there are no Gregory trees in $\mathbf{V}[G]$, it is sufficient to show in $\mathbf{V}$ that whenever $\mathbb{1}$ forces $\mathring{\mathbb{T}}$ to be a subtree of $2^{<\omega_1}$ with the CTP, $\mathbb{1}$ also forces $\mathring{\mathbb{T}}$ to have a cofinal branch. By the CTP, $|\mathbb{T}| = 2^{\aleph_0} = \kappa$ in $\mathbf{V}[G]$, so as noted above, $\mathbb{T}$ has a name of the form $\mathring{\mathbb{T}} = \{\langle \mathring{b}_\mu, \mathbb{1}\rangle : \mu < \kappa\}$, where each $\mathring{b}_\mu$ is a nice name for a subset of some $\xi_\mu \times 2$, where $\xi_\mu < \omega_1$ and $\mathbb{1} \Vdash \mathring{b}_\mu \in 2^{<\omega_1} \wedge \mathrm{ht}(\mathring{b}_\mu) \leq \xi_\mu$.

We must specify our $\diamondsuit$ sequence before we have defined an order on the sets $\mathbb{P}_\alpha = \delta_\alpha$. The definition of $\check{x}$ only uses the identity $\mathbb{1} = 0$, but the notion of "nice name" presupposes that we know what an antichain is. So, call $\mathring{b}$ a *pseudo-nice* δ–name for a subset of $X \in V$ iff $\mathring{b} = \bigcup\{\{\check{x}\} \times E_x : x \in X\}$, where each $E_x \in [\delta]^{\leq\omega}$. Then every nice name using the eventual order on δ_α will be also pseudo-nice.

Our $\diamondsuit$ sequence will make believe that $\delta_\alpha = \alpha$, since this is true on a club. So, for $\alpha \in S = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega_1\}$, choose a $\mathring{\mathbb{T}}_\alpha$ of the form $\{\langle \mathring{b}^\alpha_\mu, \mathbb{1}\rangle : \mu < \alpha\}$, where each $\mathring{b}^\alpha_\mu$ is a pseudo-nice α–name for a subset of some $\xi^\alpha_\mu \times 2$, where $\xi^\alpha_\mu < \omega_1$. These $\mathring{\mathbb{T}}_\alpha$ must have the $\diamondsuit$ property that whenever $\mathring{\mathbb{T}} = \{\langle \mathring{b}_\mu, \mathbb{1}\rangle : \mu < \kappa\}$ has the analogous form (replacing α with κ), the set of $\alpha \in S$ for which $\mathring{\mathbb{T}}_\alpha = \{\langle \mathring{b}_\mu, \mathbb{1}\rangle : \mu < \alpha\}$ is stationary.

Now, when $\alpha \in S$ and we have constructed $\mathbb{P}_\alpha$ (i.e., we know the ordinal δ_α and its ordering), choose $\mathring{\mathbb{Q}}_\alpha$ as follows: $\mathring{\mathbb{Q}}_\alpha$ is a name for the trivial one-element order *unless* $\delta_\alpha = \alpha$ and each $\mathring{b}^\alpha_\mu$ is indeed a nice $\mathbb{P}_\alpha$–name and $\mathbb{1} \Vdash_\alpha \mathring{b}^\alpha_\mu \in 2^{<\omega_1}$ and $\mathbb{1} \Vdash_\alpha$ "$\mathring{\mathbb{T}}_\alpha$ is a subtree of $2^{<\omega_1}$ with the CTP". In that case, $\mathbf{V}_\alpha$ will contain the tree $\mathbb{T}_\alpha$ which (in $\mathbf{V}_\alpha$) has the CTP, and then Theorem 4.7 applies to construct a Suslin subtree $\mathbb{Q}_\alpha \subset \mathbb{T}_\alpha$. Then, back in $\mathbf{V}$, we let $\mathring{\mathbb{Q}}_\alpha$ be a name for this $\mathbb{Q}_\alpha$, so that in $\mathbf{V}_{\alpha+1}$ we have a cofinal branch in $\mathbb{Q}_\alpha$.

Finally to show that there are no Gregory trees in $\mathbf{V}[G]$, assume that $\mathbb{1}$ forces $\mathring{\mathbb{T}}$ to be a subtree of $2^{<\omega_1}$ with the CTP. Let the $\mathring{\mathbb{T}}^\alpha$ be as in Lemma 4.3. Then, by Lemma 4.3, there is then a club C of ω_1–limits such that for $\alpha \in C$: $\delta_\alpha = \alpha$, $\mathring{\mathbb{T}}^\alpha$ is a $\mathbb{P}_\alpha$–name, and $\mathbb{1} \Vdash_\alpha$ "$\mathring{\mathbb{T}}^\alpha$ is a subtree of $2^{<\omega_1}$ with the CTP". Choosing $\alpha \in C$ with $\mathring{\mathbb{T}}^\alpha = \mathring{\mathbb{T}}_\alpha$ shows that $\mathbb{1}$ forces that there is (in $V_{\alpha+1}$) a cofinal branch in $\mathring{\mathbb{T}}$. $\dashv$

REFERENCES

[1] J. Gregory, *A countably distributive complete Boolean algebra not uncountably representable*, Proc. Amer. Math. Soc. **42** (1974), 42–46.

[2] J. Gregory, *Higher Souslin trees and the generalized continuum hypothesis*, J. Symbolic Logic **41** (1976), 663–671.

[3] J. Hart and K. Kunen, *Inverse limits and function algebras*, Topology Proceedings **30** (2006), 501–521.

[4] J. Hart and K. Kunen, *First countable continua and proper forcing*, Canad. J. Math. (to appear).

[5] K. Kunen, *Set Theory*, North–Holland, 1980.

[6] J. T. Moore, M. Hrušák, and M. Džamonja, *Parametrized $\diamondsuit$ principles*, Trans. Amer. Math. Soc. **356** (2004), 2281–2306.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
MADISON, WI 57306 USA
E-mail: kunen@math.wisc.edu
URL: http://www.math.wisc.edu/~kunen

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
MADISON, WI 57306 USA
E-mail: raghavan@math.wisc.edu
URL: http://www.math.wisc.edu/~raghavan