GALVIN'S PROBLEM IN HIGHER DIMENSIONS

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ABSTRACT. It is proved that for each natural number n, if $|\mathbb{R}| = \aleph_n$, then there is a coloring of $[\mathbb{R}]^{n+2}$ into $\aleph_0$ colors that takes all colors on $[X]^{n+2}$ whenever X is any set of reals which is homeomorphic to $\mathbb{Q}$. This generalizes a theorem of Baumgartner and sheds further light on a problem of Galvin from the 1970s. Our result also complements and contrasts with our earlier result saying that any coloring of $[\mathbb{R}]^2$ into finitely many colors can be reduced to at most 2 colors on the pairs of some set of reals which is homeomorphic to $\mathbb{Q}$ when large cardinals exist.

1. INTRODUCTION

One of the fundamental results of combinatorics is Ramsey's theorem [11], which says that every function from $[\omega]^2$ to a finite set is constant on $[A]^2$ for some infinite $A \subseteq \omega$. In 1933 Sierpiński [12] found a function from $[\mathbb{R}]^2$ into 2 which is not constant on $[A]^2$ for any infinite $A \subseteq \mathbb{R}$ that is dense in itself or with the induced ordering from $\mathbb{R}$ is isomorphic to $\langle \mathbb{Q}, < \rangle$. An unpublished result of Galvin (see p.275 of [4]) showed that for every function c from $[\mathbb{Q}]^2$ to a finite set, there exists $A \subseteq \mathbb{Q}$ such that $\langle A, < \rangle$ is isomorphic to $\langle \mathbb{Q}, < \rangle$ and c takes at most 2 values on $[A]^2$. A consequence of Galvin's work is that Sierpiński's coloring is, in a certain sense, the most complicated coloring of the unordered pairs of the structure $\langle \mathbb{Q}, < \rangle$ into finitely many colors. In an unpublished work Laver (see p.275 of [4]) generalized Galvin's result by showing that for each natural number $k \geq 1$, there exists $t_k \in \omega$ such that for every function c from $[\mathbb{Q}]^k$ to a finite set, there exists $A \subseteq \mathbb{Q}$ such that c attains at most t_k values on $[A]^k$. The claim that the optimal value of t_k is $k!(k-1)!$ (see p.275 of [4]) was later corrected by Devlin [2] to $\tan^{2k-1}(0)$ (see also Section 6.3 of [15]).

Answering a problem of van Douwen motivated by Galvin's theorem about the ordered copies of $\mathbb{Q}$ mentioned above, Baumgartner was the first to prove that there is a significant difference in the Ramsey properties of the linear order $\langle \mathbb{Q}, < \rangle$ and the topological space $\langle \mathbb{Q}, \mathcal{T}_\mathbb{Q} \rangle$, where $\mathcal{T}_\mathbb{Q}$ denotes the usual topology on $\mathbb{Q}$ inherited from $\mathbb{R}$. In [1], he found a coloring of $[\mathbb{Q}]^2$ into infinitely many colors so that no color is omitted on $[A]^2$, for any $A \subseteq \mathbb{Q}$ that is homeomorphic to $\mathbb{Q}$. Todorcevic [14] showed how to get the rectangular version of Baumgartner's result. In other words, he produced a $c : [\mathbb{Q}]^2 \to \omega$ such that no color is omitted on $A \circledast B$, whenever $A, B \subseteq \mathbb{Q}$ are both homeomorphic to $\mathbb{Q}$. Here, the notation $A \circledast B$ denotes $\{\{x, y\} : x \in A, y \in B, \text{ and } x \neq y\}$. Thus, in order to have an analogue of Galvin's

result for topological copies of $\mathbb{Q}$ instead of ordered ones, one needs to color un-
ordered pairs of some larger space such as, for example, $\mathbb{R}$. This was, in fact, the
statement of a well-known problem of Galvin popularized in the 1970s but first
appearing in print in [1]. Recently, Raghavan and Todorcevic [9] have verified this
conjecture of Galvin by proving that there is no analogue of these strong colorings
found by Baumgartner [1] and by Todorcevic [14] for $[\mathbb{R}]^2$, at least in the presence of
large cardinals. More precisely, they showed that if there is a Woodin cardinal, and
if X is an uncountable set of reals, then every coloring of $[X]^2$ into finitely many
colors attains at most 2 colors on $[A]^2$, for some $A \subseteq X$ that is homeomorphic to
$\mathbb{Q}$. It should be noted here that by the work of Todorcevic [13], there is a coloring
of $[\omega_1]^2$ into ω_1 colors such that no color is omitted on $[A]^2$, for any uncountable
set $A \subseteq \omega_1$. A consequence of Raghavan and Todorcevic's result is that Sierpiński's
coloring is, in a certain sense, the most complicated coloring of $[\mathbb{R}]^2$ into finitely
many colors when restriction to arbitrary dense in itself subspaces of $\mathbb{R}$ is allowed.
In this paper, we examine the higher-dimensional versions of Galvin's problem,
or in other words, the possibility of having analogues of Laver's result mentioned
above for topological rather than order-isomorphic copes of $\mathbb{Q}$. The main result of
this paper is that it is impossible to directly generalize the results of [9] to $[\mathbb{R}]^k$,
for $k \geq 3$ and that the cardinality of $\mathbb{R}$ now becomes relevant. More precisely, we
will prove below that if X is a set of reals of size $\aleph_n$, then there is a coloring of
$[X]^{n+2}$ into infinitely many colors such that no color is omitted on $[A]^2$, for any
$A \subseteq X$ that is homeomorphic to $\mathbb{Q}$. Therefore there are complicated colorings that
are essentially different from Sierpiński's coloring in higher dimensions. Moreover,
this shows that if we are to have finite Ramsey degrees of topological copies of $\mathbb{Q}$
in $\mathbb{R}$ for all finite dimensions simultaneously, then the continuum has to be at least
$\aleph_{\omega+1}$. Whether $\aleph_{\omega+1}$ is indeed the minimal possible value of the continuum allow-
ing finite Ramsey degrees of topological copies of $\mathbb{Q}$ inside $\mathbb{R}$ in all finite dimensions
simultaneously is the most interesting open problem left out by our work.

2. Ramsey Degrees

The results mentioned in the introduction are part of a more general framework.
We discuss this more general framework in this section, which will allow us to place
our results within the context of a broader research program in Ramsey theory.
The terminology and notation introduced in this section will also be helpful with
making the statements more succinct.

Let A and B be structures. For natural numbers $k, l, t \geq 1$, the notation

$$B \rightarrow (A)^k_{l,t}$$

means that for every coloring $c : [B]^k \rightarrow l$, there exists a substructure C of B
such that C is isomorphic to A and $\left| c''[C]^k \right| \leq t$. Suppose that $\mathcal{C}$ is some class of
structures and that A is a structure that embeds into every member of $\mathcal{C}$. For a
natural number $k \geq 1$, the k-dimensional Ramsey degree of A within $\mathcal{C}$ is the the
smallest natural number $t_k \geq 1$ (if it exists) such that $B \rightarrow (A)^k_{l,t_k}$, for every natural
number $l \geq 1$ and for every structure $B \in \mathcal{C}$. When no such t_k exists, we say that
the k-dimensional Ramsey degree of A within $\mathcal{C}$ is infinite or does not exist. Thus,
in this terminology, the result of Laver and Devlin discussed in the introduction
is expressed by saying that the k-dimensional Ramsey degree of $\langle \mathbb{Q}, < \rangle$ within the
class of all non-empty dense linear orders without endpoints exists for every $k \geq 1$
and it takes the value given by the formula $t_k = \tan^{2k-1}(0)$ of Devlin [2].

Suppose that $R_1, \ldots, R_m$ are finitely many finitary relations on the structure A.
The relations $R_1, \ldots, R_m$ are said to solve the expansion problem for A within the

class $\mathcal{C}$ if for every structure $B \in \mathcal{C}$ and every finitary relation S on B, there exists a substructure C of B and an isomorphism $\varphi : A \to C$ such that the restriction of S to C is quantifier free definable from the images of $R_1, \ldots, R_m$ under φ. It turns out that solving the expansion problem for A within $\mathcal{C}$ for k-ary relations is equivalent to finding the k-dimensional Ramsey degree of A within $\mathcal{C}$.

The notion of k-dimensional Ramsey degree discussed here is related to, but is distinct from other notions of Ramsey degree occurring frequently in the literature, such as in [6, 8, 17]. The computation of Ramsey degrees and the accompanying solution to expansion problems plays a crucial role in topological dynamics under the guise of determining the universal minimal flows of various automorphism groups. We refer to [6] for further details.

The focus of our work is on determining the k-dimensional Ramsey degrees of the topological space of the rationals within various different classes of Hausdorff spaces. Let $\mathcal{T}_{\mathbb{R}}$ denote the usual topology of the real numbers, and $\mathcal{T}_{\mathbb{Q}}$ its restriction to the rationals. In the 1970s, Galvin conjectured that the 2-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ within $\{\langle \mathbb{R}, \mathcal{T}_{\mathbb{R}} \rangle\}$ is 2. As noted in the introduction, Galvin's conjecture was brought into sharper focus by Baumgartner who proved that the 2-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ does not exist within any class which contains a countable Hausdorff space.

Theorem 1 (Baumgartner [1]). *Suppose $\langle X, \mathcal{T} \rangle$ is any Hausdorff space with $|X| = \aleph_0$. There is a coloring $c : [X]^2 \to \omega$ such that for any subspace $R \subseteq X$ that is homeomorphic to $\mathbb{Q}$, $c''[R]^2 = \omega$.*

It should be noted that Baumgartner's Theorem 1 does not exclude the possibility of positive Ramsey type results on the topological space $\mathbb{Q}$ if further restrictions are placed on the coloring. For example, Todorcevic [15] showed that every continuous coloring of $[\mathbb{Q}]^2$ into finitely many colors is constant on $[X]^2$, for some $X \subseteq \mathbb{Q}$ that is homeomorphic to $\mathbb{Q}$. Another example is a recent result of Raghavan and Todorcevic [10] who showed that if $\{x_n : n \in \omega\}$ is any enumeration of $\mathbb{Q}$ and $c : [\omega]^2 \to \omega$ is a function which satisfies certain ultrametric inequalities, then there is a $B \subseteq \omega$ such that $\{x_n : n \in B\}$ is homeomorphic to $\mathbb{Q}$ and c is shift-increasing on B.

An unpublished result of Todorcevic and Weiss extended Baumgartner's result to include σ-discrete metric spaces.

Theorem 2 (Todorcevic and Weiss [16]). *If $\langle X, d \rangle$ is a σ-discrete metric space, then there is a coloring $c : [X]^2 \to \omega$ such that $c''[Y]^2 = \omega$ for all $Y \subseteq X$ homeomorphic to $\mathbb{Q}$.*

In 2018, the authors proved in [9] that a σ-discrete metric space is the only impediment which can prevent the 2-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ from being equal to 2 within a class of metric spaces. The following summarizes the main results from [9].

Theorem 3 ([9]). *If there is a Woodin cardinal, then the 2-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ within the class of all uncountable sets of reals is 2. If there is a proper class of Woodin cardinals, then the 2-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ within the class of all non-σ-discrete metric spaces is equal to 2, and it is at most 2 within the class of all regular, non-left-separated spaces with point-countable bases. Further, these conclusions hold under PFA.*

Theorem 3 leads to a simple solution of the expansion problem in the case of binary relations for $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ within the class of metric spaces. In fact for uncountable sets of real numbers, every binary relation is quantifier free definable from equality,

the usual ordering of the reals, and an arbitrary well-ordering on a homeomorphic copy of $\mathbb{Q}$. The large cardinal hypothesis of Theorem 3 can be weakened to a local statement on the existence of inner models that contain Woodin cardinals and correctly compute sufficiently large fragments of the cumulative hierarchy. We refer to [9] for further details.

The main result of this paper is that there is no direct generalization of Theorem 3 to dimensions 3 and higher. We will prove that if $\langle X, \mathcal{T} \rangle$ is any Hausdorff space of cardinality $\aleph_n$, then there is a coloring of $[X]^{n+2}$ into $\aleph_0$ colors which realizes all colors on $[R]^{n+2}$ for any subspace $R \subseteq X$ that is homeomorphic to $\mathbb{Q}$. In other words, if a class $\mathcal{C}$ contains any Hausdorff space of size $\aleph_n$, then the $n+2$-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ within $\mathcal{C}$ does not exist. In particular, if CH holds, then the 3-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ within $\{\langle \mathbb{R}, \mathcal{T}_{\mathbb{R}} \rangle\}$ does not exist, and if the k-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ does exist within $\{\langle \mathbb{R}, \mathcal{T}_{\mathbb{R}} \rangle\}$ for every k, then the continuum must be at least $\aleph_{\omega+1}$. Our result here shows that Theorem 3 is sharp at least as far as the dimension is concerned.

Sierpiński's coloring from [12] mentioned in the introduction shows that the k-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ within $\{\langle \mathbb{R}, \mathcal{T}_{\mathbb{R}} \rangle\}$ is at least $k!(k-1)!$. There is also a natural way to generalize Sierpiński's coloring to any metrizable space. Details may be found in Section 3 of [9]. In view of this and the result we prove here, the following question suggests itself.

Question 4. *What is the largest class of topological spaces within which the k-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ is equal to $k!(k-1)!$?*

3. NOTATION

Our set-theoretic notation is standard. If c is a function, then $\mathrm{dom}(c)$ denotes the domain of c, and if $X \subseteq \mathrm{dom}(c)$, then $c''X$ is the image of X under c – that is, $c''X = \{c(x) : x \in X\}$. For any A, $\mathcal{P}(A)$ denotes the powerset of A. When κ is a cardinal, $[X]^\kappa$ is $\{A \subseteq X : |A| = \kappa\}$, and $[X]^{<\kappa}$ denotes $\{A \subseteq X : |A| < \kappa\}$.

If d is a metric on Y, then $B_d(y, \epsilon)$ denotes $\{z \in Y : d(y, z) < \epsilon\}$, for all $y \in Y$ and $\epsilon \in \mathbb{R}$. A topological space $\langle X, \mathcal{T} \rangle$ is *dense-in-itself* if for each $x \in X$ and each open neighborhood U of x, there exists $y \in U$ with $y \neq x$. A theorem of Sierpiński (see [3]) says that $\langle X, \mathcal{T} \rangle$ is homeomorphic to $\mathbb{Q}$ with its usual topology if and only if it is non-empty, countable, metrizable, and dense-in-itself.

4. THE MAIN RESULT

We prove the main result in this section. The proof is a natural generalization of Baumgartner's proof from [1] via the following well-known set mapping theorem of Kuratowski [7], which is usually formulated in a slightly different way. Much more information about set mappings in general, and Kuratowski's theorem in particular, may be found in [5].

Lemma 5 (Kuratowski [7]). *For each $n \in \omega$, there exists $f_n : [\omega_n]^{n+1} \to [\omega_n]^{<\aleph_0}$ such that:*

(1) $\forall s \in [\omega_n]^{n+1} [f_n(s) \subseteq \max(s)]$;
(2) $\forall t \in [\omega_n]^{n+2} \exists \alpha \in t [\alpha < \max(t) \text{ and } \alpha \in f_n(t \setminus \{\alpha\})]$.

Proof. The proof is by induction on $n \in \omega$. For $n = 0$, define $f_0 : [\omega]^1 \to [\omega]^{<\aleph_0}$ by $f_0(\{m\}) = m \subseteq m = \max(\{m\})$. For (2), if $\{l, m\} \in [\omega]^2$ with $l < m$, then $l \in m = f_0(\{m\}) = f_0(\{l, m\} \setminus \{l\})$. Now assume that $f_n : [\omega_n]^{n+1} \to [\omega_n]^{<\aleph_0}$ satisfying (1) and (2) is given. Fix a sequence $\langle e_\gamma : \gamma \in \omega_{n+1} \rangle$ such that for each $\gamma \in \omega_{n+1}$ $e_\gamma : \gamma \to \omega_n$ is 1-1. Define $f_{n+1} : [\omega_{n+1}]^{n+2} \to [\omega_{n+1}]^{<\aleph_0}$ as follows. Given

$s \in [\omega_{n+1}]^{n+2}$, let $\gamma = \max(s) \in s \subseteq \omega_{n+1}$. Then $s \setminus \{\gamma\} \subseteq \gamma$ and $e_\gamma''(s \setminus \{\gamma\}) \in [\omega_n]^{n+1}$. Thus we may define $f_{n+1}(s) = e_\gamma^{-1}\left(f_n\left(e_\gamma''(s \setminus \{\gamma\})\right)\right) \subseteq \gamma = \max(s)$. To verify (2), let $t \in [\omega_{n+1}]^{n+3}$ be given. Let $\gamma = \max(t) \in t \subseteq \omega_{n+1}$. Then $t \setminus \{\gamma\} \subseteq \gamma$ and $u = e_\gamma''(t \setminus \{\gamma\}) \in [\omega_n]^{n+2}$. By the induction hypothesis, there exists $\alpha' \in u$ such that $\alpha' \in f_n(u \setminus \{\alpha'\})$. Let $\alpha \in t \setminus \{\gamma\}$ be so that $e_\gamma(\alpha) = \alpha'$, and observe that $\alpha \in e_\gamma^{-1}(f_n(u \setminus \{\alpha'\}))$. Note that $\alpha \in t$ and that $\alpha < \gamma = \max(t)$. Put $s = t \setminus \{\alpha\}$ and note that $\max(s) = \gamma$. Since $t \setminus \{\gamma\} \subseteq \gamma$, $\{\alpha\} \subseteq \gamma$, and e_γ is a 1-1 function with domain γ, $e_\gamma''(s \setminus \{\gamma\}) = e_\gamma''((t \setminus \{\gamma\}) \setminus \{\alpha\}) = e_\gamma''(t \setminus \{\gamma\}) \setminus e_\gamma''\{\alpha\} = u \setminus \{\alpha'\}$. Therefore by definition, $f_{n+1}(t \setminus \{\alpha\}) = f_{n+1}(s) = e_\gamma^{-1}\left(f_n\left(e_\gamma''(s \setminus \{\gamma\})\right)\right) = e_\gamma^{-1}(f_n(u \setminus \{\alpha'\})) \ni \alpha$. $\dashv$

The next lemma says that when searching within a well-ordered topological space for a homeomorphic copy of $\mathbb{Q}$ on which some coloring is nicely behaved, one can concentrate only on the copies of $\mathbb{Q}$ which have order type ω with respect to the well-ordering. This lemma has been applied in proving the main result of [10].

Lemma 6. *Let $\langle X, \mathcal{T} \rangle$ be a topological space and δ an ordinal. Suppose that $\langle x_\alpha : \alpha < \delta \rangle$ is a 1-1 enumeration of all the points of X. Let $\mathcal{P} \subseteq [\delta]^{<\aleph_1}$ be a family such that:*

(1) $\mathcal{P}$ is hereditary, that is, $\forall A \in \mathcal{P} \forall B \subseteq A\, [B \in \mathcal{P}]$;

(2) there exists $A \in \mathcal{P}$ such that the subspace $\{x_\alpha : \alpha \in A\}$ is homeomorphic to $\mathbb{Q}$.

Then

$$\omega = \min\{\mathrm{otp}(A) : A \in \mathcal{P} \text{ and the subspace } \{x_\alpha : \alpha \in A\} \text{ is homeomorphic to } \mathbb{Q}\}.$$

Proof. The right hand side of the equation being asserted is well-defined due to item (2) of the hypotheses. Let ξ denote the right hand side. Obviously, $\omega \leq \xi < \omega_1$. Fix $A \in \mathcal{P}$ such that the subspace $\{x_\alpha : \alpha \in A\}$ is homeomorphic to $\mathbb{Q}$ and $\xi = \mathrm{otp}(A)$. Recall that if $Y \subseteq X$ is a subspace which is homeomorphic to $\mathbb{Q}$, and if $U \subseteq Y$ is open in Y and if $U \neq \emptyset$, then U is homeomorphic to $\mathbb{Q}$. Let us apply this observation to $Y = \{x_\alpha : \alpha \in A\}$. Thus for every $U \subseteq Y$ which is open in Y and is non-empty, $B = \{\alpha < \delta : x_\alpha \in U\} \subseteq A$, which means $B \in \mathcal{P}$ because $\mathcal{P}$ is hereditary, and $\{x_\alpha : \alpha \in B\} = U$ is a subspace of X that is homeomorphic to $\mathbb{Q}$, which means $\mathrm{otp}(B) = \xi$ because of the minimality of ξ. Now, fix a metric d on Y that is compatible with the subspace topology on Y. It follows that for any $\alpha \in A$ and non-empty open subset U of Y, $\{\zeta < \delta : x_\zeta \in U\} \not\subseteq A \cap \alpha$, and so if $x_\alpha \notin U$, then $\exists \alpha < \beta < \delta\, [x_\beta \in U \text{ and } \beta \in A]$. Let $\{\langle n_k, l_k \rangle : 2 \leq k < \omega\}$ be a 1-1 enumeration of $\omega \times \omega$ so that $n_k < k - 1$, for all $k < \omega$. Choose $\alpha_0, \alpha_1 \in A$ with $\alpha_0 < \alpha_1$. For $2 \leq k < \omega$, suppose that $\alpha_0 < \cdots < \alpha_{k-1}$ belonging to A are given. As $n_k < k - 1$, $\alpha_{n_k} < \alpha_{k-1}$, and so $x_{\alpha_{n_k}} \neq x_{\alpha_{k-1}}$. Let $\epsilon > 0$ be such that $B_d\left(x_{\alpha_{n_k}}, \epsilon\right) \subseteq B_d\left(x_{\alpha_{n_k}}, \frac{1}{l_k+1}\right)$ and $x_{\alpha_{k-1}} \notin B_d\left(x_{\alpha_{n_k}}, \epsilon\right)$. Then there exists $\alpha_{k-1} < \alpha_k < \delta$ such that $x_{\alpha_k} \in B_d\left(x_{\alpha_{n_k}}, \epsilon\right)$ and $\alpha_k \in A$. Now $B = \{\alpha_k : k < \omega\} \subseteq A$ and $\mathrm{otp}(B) = \omega$ by construction. Since $\mathcal{P}$ is hereditary, $B \in \mathcal{P}$. Also $\{x_\alpha : \alpha \in B\}$, being a subspace of Y, is countable, metrizable, non-empty, and by construction, it is dense-in-itself. Therefore, the subspace $\{x_\alpha : \alpha \in B\}$ is homeomorphic to $\mathbb{Q}$, and therefore $\xi = \omega$. $\dashv$

Theorem 7. *Let $n \in \omega$. Let $\langle X, \mathcal{T} \rangle$ be any Hausdorff space with $|X| = \aleph_n$. There is a coloring $c : [X]^{n+2} \to \omega$ such that for any subspace $R \subseteq X$ that is homeomorphic to $\mathbb{Q}$, $c''[R]^{n+2} = \omega$.*

Proof. Fix $f_n : [\omega_n]^{n+1} \to [\omega_n]^{<\aleph_0}$ satisfying (1) and (2) of Lemma 5. Fix also a 1–1 enumeration $\langle y_\alpha : \alpha < \omega_n \rangle$ of all points of X. For any $s \in [\omega_n]^{n+1}$, define $O(s) = f_n(s) \setminus s$. By (1) of Lemma 5, $O(s)$ is a finite subset of $\max(s)$. Since $\langle X, \mathcal{T} \rangle$ is Hausdorff, it is possible to find $\{N_s(y_\xi) : \xi \in O(s) \cup s\}$ such that $N_s(y_\xi)$ is an open neighborhood of y_ξ, and for all $\xi \neq \xi'$ belonging to $O(s) \cup s$, $N_s(y_\xi) \cap N_s(y_{\xi'}) = \emptyset$. Observe that if $t \in [\omega_n]^{n+2}$ and if there exists some $\xi \in O(t \setminus \{\max(t)\})$ such that $y_{\max(t)} \in N_{t \setminus \{\max(t)\}}(y_\xi)$, then such ξ must be unique.

Define a function $h : [\omega_n]^{n+2} \cup \{\uparrow\} \to [\omega_n]^{n+2} \cup \{\uparrow\}$ as follows. First define $h(\uparrow) = \uparrow$. Next for any $t \in [\omega_n]^{n+2}$, define

$$h(t) = \begin{cases} (t \setminus \{\max(t)\}) \cup \{\xi\} & \text{if } \xi \in O(t \setminus \{\max(t)\}) \text{ and} \\ & y_{\max(t)} \in N_{t \setminus \{\max(t)\}}(y_\xi), \\ \\ & \text{if there is no } \xi \in O(t \setminus \{\max(t)\}) \\ \uparrow & \text{such that } y_{\max(t)} \in N_{t \setminus \{\max(t)\}}(y_\xi). \end{cases}$$

First, note that since $O(t \setminus \{\max(t)\}) \cap (t \setminus \{\max(t)\}) = \emptyset$, $\xi \notin (t \setminus \{\max(t)\})$, and so $(t \setminus \{\max(t)\}) \cup \{\xi\}$ does indeed belong to $[\omega_n]^{n+2}$. Next, note that since $O(t \setminus \{\max(t)\}) \subseteq \max(t \setminus \{\max(t)\})$, $\max((t \setminus \{\max(t)\}) \cup \{\xi\}) < \max(t)$. Now for any $t \in [\omega_n]^{n+2} \cup \{\uparrow\}$, define $h^0(t) = t$, and $h^{k+1}(t) = h(h^k(t))$, for all $k \in \omega$. As observed earlier, when $h^{k+1}(t) \neq \uparrow$, then $\max(h^{k+1}(t)) < \max(h^k(t))$. It follows that for some $k \in \omega$, $h^k(t) = \uparrow$. Since $h^0(t) \neq \uparrow$ when $t \in [\omega_n]^{n+2}$, we may define $c : [X]^{n+2} \to \omega$ as follows. For any $w \in [X]^{n+2}$ there exists a unique $t_w \in [\omega_n]^{n+2}$ with $w = \{y_\alpha : \alpha \in t_w\}$. Define

$$c(w) = \min\left(\{k \in \omega : h^{k+1}(t_w) = \uparrow\}\right).$$

We will verify that c has the required property. Towards a contradiction, assume that there exists a subspace $R \subseteq X$ that is homeomorphic to $\mathbb{Q}$ and $c''[R]^{n+2} \neq \omega$. Let $\mathcal{P} = \left\{ M \in [\omega_n]^{<\aleph_1} : c''[\{y_\alpha : \alpha \in M\}]^{n+2} \neq \omega \right\}$. If $M \in \mathcal{P}$ and $L \subseteq M$, then $[\{y_\alpha : \alpha \in L\}]^{n+2} \subseteq [\{y_\alpha : \alpha \in M\}]^{n+2}$ and so $c''[\{y_\alpha : \alpha \in L\}]^{n+2} \neq \omega$. Therefore $\mathcal{P}$ is hereditary and by hypothesis item (2) of Lemma 6 is satisfied. Hence by Lemma 6, there exists $M \in \mathcal{P}$ with $\text{otp}(M) = \omega$ so that the subspace $\{y_\alpha : \alpha \in M\}$ is homeomorphic to $\mathbb{Q}$.

We will derive a contradiction to the choice of M by showing that for each $k \in \omega$, there exists $w \in [\{y_\alpha : \alpha \in M\}]^{n+2}$ with $c(w) = k$. For $k = 0$, fix any $s \in [M]^{n+1}$. By definition, $N_s(y_{\max(s)})$ is an open neighborhood of $y_{\max(s)}$. Since the subspace $\{y_\alpha : \alpha \in M\} \subseteq X$ is homeomorphic to $\mathbb{Q}$ and since $N_s(y_{\max(s)}) \cap \{y_\alpha : \alpha \in M\}$ is a non-empty open subset of $\{y_\alpha : \alpha \in M\}$, it must be infinite. Hence we may choose $\delta \in M$ such that $\max(s) < \delta$ and $y_\delta \in N_s(y_{\max(s)})$. Let $t = s \cup \{\delta\} \in [M]^{n+2}$. Note that $\max(t) = \delta$ and $t \setminus \{\max(t)\} = s$. Further, if $\xi \in O(s)$, then $\xi \neq \max(s)$ and so $N_s(y_\xi) \cap N_s(y_{\max(s)}) = \emptyset$, whence $y_\delta \notin N_s(y_\xi)$. Therefore, $h(t) = \uparrow$. It follows that $c(w) = 0$, where $w = \{y_\alpha : \alpha \in t\}$.

Proceeding by induction, suppose that $k \in \omega$ and that for some $u \in [M]^{n+2}$, $c(\{y_\alpha : \alpha \in u\}) = k$. This means that $h^{k+1}(u) = \uparrow$, but $h^k(u) \neq \uparrow$. By (2) of Lemma 5, there exists $\alpha \in u$ with $\alpha \in f_n(u \setminus \{\alpha\})$. Define $s = u \setminus \{\alpha\} \in [M]^{n+1} \subseteq [\omega_n]^{n+1}$. Note that $\alpha \in f_n(s) \setminus s = O(s)$. By definition, $N_s(y_\alpha)$ is an open neighborhood of y_α. Hence $y_\alpha \in N_s(y_\alpha) \cap \{y_\beta : \beta \in M\}$, and so $N_s(y_\alpha) \cap \{y_\beta : \beta \in M\}$ is a non-empty open subset of the subspace $\{y_\beta : \beta \in M\} \subseteq X$, which is homeomorphic to $\mathbb{Q}$. Therefore $N_s(y_\alpha) \cap \{y_\beta : \beta \in M\}$ is an infinite set. Choose $\delta \in M$ such that $\delta > \max(s)$ and $y_\delta \in N_s(y_\alpha)$. Define $t = s \cup \{\delta\} \in [M]^{n+2}$. Note that $\max(t) = \delta$ and that $t \setminus \{\max(t)\} = s$. Therefore by definition, $h(t) = s \cup \{\alpha\} = (u \setminus \{\alpha\}) \cup \{\alpha\} = u$

because $\{\alpha\} \subseteq u$. It follows that $h^{k+2}(t) = \uparrow$, but $h^{k+1}(t) \neq \uparrow$, and hence that $c(\{y_\beta : \beta \in t\}) = k + 1$. This concludes the induction and the proof. ⊣

Observe that Baumgartner's Theorem 1 is simply the special case of Theorem 7 when $n = 0$. We will now point out a few corollaries of Theorem 7.

Corollary 8. *Let* $n \in \omega$. *Let* $\langle X, \mathcal{T} \rangle$ *be any Hausdorff space with* $|X| = \aleph_n$. *For each natural number* $l \geq 1$, *there is a coloring* $d : [X]^{n+2} \to l$ *such that for any subspace* $R \subseteq X$ *that is homeomorphic to* $\mathbb{Q}$, $d''[R]^{n+2} = l$.

Proof. Let $c : [X]^{n+2} \to \omega$ be the coloring from Theorem 7. Define $d : [X]^{n+2} \to l$ so that for each $w \in [X]^{n+2}$, $d(w) = c(w) \mod l$. So $d(w) \in \{0, \ldots, l-1\}$. Now apply Theorem 7. ⊣

Corollary 9. *Let* $n \in \omega$. *Suppose* $\mathcal{C}$ *is any class of topological spaces. If* $\mathcal{C}$ *contains any Hausdorff space of cardinality at most* $\aleph_n$, *then the* $n + 2$-*dimensional Ramsey degree of* $\langle \mathbb{Q}, \mathcal{T}_\mathbb{Q} \rangle$ *within* $\mathcal{C}$ *does not exist.*

Corollary 9 says that in ZFC the 3-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_\mathbb{Q} \rangle$ within the class of all sets of reals of cardinality $\aleph_1$ does not exist. Thus the partition relation $\langle \mathbb{R}, \mathcal{T}_\mathbb{R} \rangle \to (\langle \mathbb{Q}, \mathcal{T}_\mathbb{Q} \rangle)^k_{l,k!(k-1)!}$ serves as a gauge of the size of the continuum. An intriguing feature here is that testing for a well-behaved countable substructure of $\mathbb{R}$ can reveal something about the size of the continuum.

Corollary 10. *If* $\langle \mathbb{R}, \mathcal{T}_\mathbb{R} \rangle \to (\langle \mathbb{Q}, \mathcal{T}_\mathbb{Q} \rangle)^3_{13,12}$, *then* CH *fails. For any* $k \geq 1$, *if for every* $1 \leq l < \omega$, $\langle \mathbb{R}, \mathcal{T}_\mathbb{R} \rangle \to (\langle \mathbb{Q}, \mathcal{T}_\mathbb{Q} \rangle)^k_{l,k!(k-1)!}$, *then* $|\mathbb{R}| \geq \aleph_{k-1}$. *If the* k-*dimensional Ramsey degree of* $\langle \mathbb{Q}, \mathcal{T}_\mathbb{Q} \rangle$ *in* $\{\langle \mathbb{R}, \mathcal{T}_\mathbb{R} \rangle\}$ *exists for every natural number* $k \geq 1$, *then* $2^{\aleph_0} \geq \aleph_{\omega+1}$.

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