BOUNDING, SPLITTING, AND ALMOST DISJOINTNESS

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Abstract. We investigate some aspects of bounding, splitting, and almost disjointness. In particular, we investigate the relationship between the bounding number, the closed almost disjointness number, the splitting number, and the existence of certain kinds of splitting families.

1. Introduction

The closed (and Borel) almost disjointness number was recently introduced by Brendle and Khomskii [BK], and has received a lot of attention. We study the connections between this number and the notions of bounding and splitting in this paper. We start with some basic definitions. Recall that two infinite subsets a and b of ω are *almost disjoint or a.d.* if $a \cap b$ is finite. We say that a family $\mathscr{A}$ of infinite subsets of ω is *almost disjoint or a.d.* if its members are pairwise almost disjoint. A *Maximal Almost Disjoint family, or MAD family* is an infinite a.d. family with the property that $\forall b \in [\omega]^{\omega} \exists a \in \mathscr{A} \, [|a \cap b| = \aleph_0]$. The cardinal invariant $\mathfrak{a}$ is the least κ such that there is a MAD family of size κ. Recall that $\mathfrak{b}$ is the least size of a subset of $\langle \omega^{\omega}, \leq^{*} \rangle$ that does not have an upper bound. It is well-known that $\mathfrak{b} \leq \mathfrak{a}$. For $x, a \in \mathcal{P}(\omega)$, x *splits* a if $|x \cap a| = |(\omega \setminus x) \cap a| = \omega$. $\mathcal{F} \subset \mathcal{P}(\omega)$ is called a *splitting family* if $\forall a \in [\omega]^{\omega} \exists x \in \mathcal{F} \, [x \text{ splits } a]$. $\mathfrak{s}$ is the least size of a splitting family. $\mathcal{F} \subset \mathcal{P}(\omega)$ is called an *ω-splitting family* if for any collection $\{a_n : n \in \omega\} \subset [\omega]^{\omega}$, there exists $x \in \mathcal{F}$ such that $\forall n \in \omega \, [x \text{ splits } a_n]$. $\mathfrak{s}_{\omega}$ is the least size of an ω-splitting family.

Brendle and Khomskii [BK] studied the possible descriptive complexities of MAD families in certain forcing extensions of $\mathbf{L}$. This led them to consider the following cardinal invariant.

Definition 1. $\mathfrak{a}_{closed}$ is the least κ such that there are κ closed subsets of $[\omega]^{\omega}$ whose union is a MAD family in $[\omega]^{\omega}$.

Obviously, $\mathfrak{a}_{closed} \leq \mathfrak{a}$. Brendle and Khomskii showed in [BK] that $\mathfrak{a}_{closed}$ behaves differently from $\mathfrak{a}$ by showing that $\mathfrak{a}_{closed} = \aleph_1 < \aleph_2 = \mathfrak{b}$ holds in the Hechler model. Heuristically, the difference between $\mathfrak{a}$ and $\mathfrak{a}_{closed}$ may be seen by considering how a witness to $\mathfrak{a}_{closed} = \aleph_1$ can be destroyed in a forcing extension. If $\mathscr{A} = \bigcup_{\alpha < \omega_1} X_{\alpha}$ is a witness to $\mathfrak{a}_{closed} = \omega_1$, where the X_{α} are closed subsets of $[\omega]^{\omega}$

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coded in the ground model, then to destroy $\mathscr{A}$ it is necessary to add a set $b \in [\omega]^{\omega}$ which is almost disjoint from every member of every X_α *even after* these codes have been reinterpreted in the forcing extension. Interpreting a ground model code in a forcing extension results in a larger set of reals. This makes increasing $\mathfrak{a}_{closed}$ harder than increasing $\mathfrak{a}$, and this fact was exploited by Brendle and Khomskii in their above mentioned result.

In Sections 2 and 4 we prove the consistency of $\mathfrak{b} < \mathfrak{a}_{closed}$. So taken together with the earlier result of Brendle and Khomskii, this establishes the mutual independence of $\mathfrak{b}$ and $\mathfrak{a}_{closed}$. Unsurprisingly, our proofs are closely modeled on the existing proofs of the consistency of $\mathfrak{b} < \mathfrak{a}$. Historically there have been two seemingly distinct methods for producing a model of $\mathfrak{b} < \mathfrak{a}$. In the first method, invented by Shelah in [Sh1], the conditions consist of a finite part followed by an infinite sequence of finite sets equipped with a measure-like structure. In the same paper, Shelah also used this method to produce the first consistency proof of $\mathfrak{b} < \mathfrak{s}$. In Section 2, we get a model of $\mathfrak{b} < \mathfrak{a}_{closed}$ using Shelah's technique. In the second method, devised by Brendle in [Br], an ultrafilter is constructed as an ascending union of F_σ filters, and then this ultrafilter is diagonalized by the corresponding Mathias-Prikry forcing. One of the byproducts of the results in this paper is that these two techniques are not so different after all. In Section 3 we show that Shelah's forcing from [Sh1] is equivalent to a two step iteration of a countably closed forcing that adds an ultrafilter which is a union of F_σ filters from the ground model succeeded by the Mathias-Prikry forcing for this generic ultrafilter. Examining this proof one quickly realizes that for the Mathias-Prikry forcing occurring in the second step of this iteration to have the right properties, it is not necessary for the ultrafilter to be fully generic with respect to the countably closed forcing occurring in the first step; it is sufficient for the ultrafilter to meet a certain collection of $\mathfrak{c}$ many dense sets. With this realization, assuming CH, it is possible to build a sufficiently generic ultrafilter in the ground model itself. In this way, we give a proof of the consistency of $\mathfrak{b} < \mathfrak{a}_{closed}$ by a finite support iteration of Mathias-Prikry forcings in Section 4 along the lines of Brendle [Br].

In Section 5 we show that the existence of certain special types of splitting families implies that $\mathfrak{a}_{closed} = \omega_1$. The existence of such special splitting families is closely related to the statement $\mathfrak{s}_\omega = \omega_1$. It is unknown whether $\mathfrak{s}_\omega = \omega_1$ implies that $\mathfrak{a}_{closed} = \omega_1$. The result in Section 5 sheds some light on this, and moreover it strengthens previous results of Raghavan and Shelah [RS], and Brendle and Khomskii [BK].

Finally in Section 6, we separate the notions of club splitting and tail splitting (see Definition 31). This answers a question from [GS3].

2. Consistency of $\aleph_1 = \mathfrak{b} < \mathfrak{a}_{closed}$

In this section we show the consistency of $\mathfrak{b} < \mathfrak{a}_{closed}$ by a creature forcing. The argument is similar to the one used by Shelah in [Sh1] and [Sh2] to show the consistency of $\mathfrak{b} < \mathfrak{a}$, though we have to do some extra work to make this argument work for $\mathfrak{a}_{closed}$. The notation and presentation in this section generally follow Abraham [Ab].

Before plunging into the details, we make some remarks about the structure of the proof. The final forcing will be a countable support (CS) iteration of proper forcings which does not add a dominating real. At any stage, a specific witness to

$\mathfrak{a}_{closed} = \omega_1$, call it $\mathscr{A}$, is dealt with. We first define a proper poset $\mathbb{P}_0$ which adds an unsplit real but does not add any dominating reals (and more; see Definition 21 and following discussion). The definition of $\mathbb{P}_0$ does not depend on $\mathscr{A}$, and it may or may not destroy $\mathscr{A}$. If it does, then we simply force with $\mathbb{P}_0$. If it does not, we first add ω_1 Cohen reals. In the resulting extension we define a proper poset $\mathbb{P}_1$ which depends on $\mathscr{A}$ and always destroys it. Under the assumption that $\mathbb{P}_0$ (as defined in the extension) still does not destroy $\mathscr{A}$, we prove that $\mathbb{P}_1$ does not add dominating reals (and more), so that we may force with $\mathbb{P}_1$ to take care of $\mathscr{A}$.

Definition 2. FIN denotes $[\omega]^{<\omega} \setminus \{0\}$. Let $x \subset \omega$. A function nor : $[x]^{<\omega} \to \omega$ is a said to be a *norm on x* if:

(1) $\forall s \in [x]^{<\omega} [\mathrm{nor}(s) > 0 \implies |s| > 1]$;
(2) $\forall s, t \in [x]^{<\omega} [s \subset t \implies \mathrm{nor}(s) \leq \mathrm{nor}(t)]$;
(3) for any $s, s_0, s_1 \in [x]^{<\omega}$ and for any $n > 0$, if $\mathrm{nor}(s) \geq n$ and $s = s_0 \cup s_1$, then there exists $i \in 2$ such that $\mathrm{nor}(s_i) \geq n - 1$.

A *creature* c is a pair $\langle s_c, \mathrm{nor}_c \rangle$ such that $s_c \in \mathrm{FIN}$ and nor_c is a norm on s_c such that $\mathrm{nor}_c(s_c) > 0$. Given creatures c and d, we write $c < d$ to mean $\max(s_c) < \min(s_d)$ and $\mathrm{nor}_c(s_c) < \mathrm{nor}_d(s_d)$.

A 0-*condition* p is a pair $\langle s^p, \langle c_n^p : n \in \omega \rangle \rangle$ such that:

(4) $s^p \in [\omega]^{<\omega}$;
(5) for each $n \in \omega$, c_n^p is a creature and $c_n^p < c_{n+1}^p$;
(6) $\forall m \in s^p \left[m < \min \left(s_{c_0^p} \right) \right]$.

Henceforth, s_n^p and nor_n^p will be used to denote $s_{c_n^p}$ and $\mathrm{nor}_{c_n^p}$ respectively. We may also omit the superscript p if it is clear from the context. For a 0-condition p, $\mathrm{int}(p) = \bigcup_{n \in \omega} s_n^p$. Given 0-conditions p and q, $q \leq p$ means:

(7) $s^q \supset s^p$ and $s^q \setminus s^p \subset \mathrm{int}(p)$;
(8) let n_0 be least such that $\forall m \geq n_0 [s_m^p \cap s^q = 0]$; there exists an interval partition $\langle i_n : n \in \omega \rangle$ of $[n_0, \infty)$ (that is, $i_0 = n_0$ and $\forall n \in \omega [i_n < i_{n+1}]$) such that $\forall n \in \omega \left[s_n^q \subset \bigcup_{m \in [i_n, i_{n+1})} s_m^p \right]$;
(9) for any $n \in \omega$, for any $t \subset s_n^q$, if $\mathrm{nor}_n^q(t) > 0$, then there is $m \in \omega$ such that $\mathrm{nor}_m^p(t \cap s_m^p) > 0$.

For 0-conditions p and q, we say $q \leq_0 p$ if $q \leq p$ and $s^p = s^q$. For $n > 0$, $q \leq_n p$ if $q \leq_0 p$ and for all $m \leq n - 1$, $c_m^q = c_m^p$.

Observe that clause (8) is equivalent to saying that for each $n \in \omega$, $s_n^q \subset \bigcup_{m \in [n_0, \infty)} s_m^p$ and $\max\{m \in [n_0, \infty) : s_n^q \cap s_m^p \neq 0\} < \min\{m \in [n_0, \infty) : s_{n+1}^q \cap s_m^p \neq 0\}$. This is sometimes useful for checking clause (8). Also, it is easy to see that $\leq$ and $\leq_n$ are transitive for all n.

Lemma 3. *Let $\langle p_n : n \in \omega \rangle$ be a sequence of 0-conditions and let $\langle k_n : n \in \omega \rangle$ be a sequence of elements of $\omega \setminus \{0\}$ such that $\forall n \in \omega [k_n < k_{n+1}]$. Assume that $p_{n+1} \leq_{k_n} p_n$. Define q as follows. $s^q = s^{p_n}$ for all n. For all $m \in [0, k_0)$, $c_m^q = c_m^{p_0}$. For each $m \in [k_n, k_{n+1})$, $c_m^q = c_m^{p_{n+1}}$. Then q is a 0-condition and for each $n \in \omega$, $q \leq_{k_n} p_n$.*

Proof. First note that for any n, $c_{k_n - 1}^q = c_{k_n - 1}^{p_n}$. So since $p_{n+1} \leq_{k_n} p_n$, $c_{k_n - 1}^q = c_{k_n - 1}^{p_n} = c_{k_n - 1}^{p_{n+1}} < c_{k_n}^{p_{n+1}} = c_{k_n}^q$. It follows that for all m, $c_m^q < c_{m+1}^q$, and so q is a 0-condition.

To check that $q \leq_{k_n} p_n$, note that $s^q = s^{p_n}$, and that for all $m \in [0, k_n)$, $c^q_m = c^{p_n}_m$. So it is enough to check clauses (8) and (9) of Definition 2. For clause (9), simply note that for any $m \in [k_n, \infty)$, there is a $l > n$ such that $c^q_m = c^{p_l}_m$ and that $p_l \leq p_n$. For clause (8) simply note that for any $m \in \omega$, there is a $p_l \leq p_n$ such that $c^q_m = c^{p_l}_m$ and $c^q_{m+1} = c^{p_l}_{m+1}$. $\dashv$

Fix $\langle X_\alpha : \alpha < \omega_1\rangle$ such that:

(1) X_α is a non-empty closed subset of $[\omega]^\omega$;
(2) $\mathscr{A} = \bigcup_{\alpha<\omega_1} X_\alpha$ is a MAD family.

We will be working with forcing extensions of the model in which the codes for the X_α live. We adopt the standing convention that when we write either "X_α" or "$\mathscr{A}$" while working inside such a model we mean the set that is gotten by interpreting the codes in that model. For each $\alpha < \omega_1$, let Y_α be the closure of X_α in $\mathcal{P}(\omega)$. Note that Y_α is compact and that $Y_\alpha \setminus X_\alpha \subset [\omega]^{<\omega}$.

Definition 4. Suppose p is a 0-condition. Define $A_p = \{s \in [\omega]^{<\omega} : \exists n \in \omega\, [\mathrm{nor}_n(s \cap s_n) > 0]\}$. Let $\mathcal{F}_p$ be the filter on ω generated by the set

$$C_p = \{\omega \setminus a : a \subset \omega \wedge \neg\exists s \in A_p\, [s \subset a]\}.$$

All filters on ω are assumed to contain the Fréchet filter. Note that C_p is a closed subset of $\mathcal{P}(\omega)$ and so $\mathcal{F}_p$ is F_σ in $\mathcal{P}(\omega)$. Note also that for any $i \in \omega$, if $\omega \setminus i \subset a_0 \cup \cdots \cup a_k$ and $n \in \omega$ is such that $i \cap s_n = 0$ and $\mathrm{nor}_n(s_n) > k + 1$, then for some $0 \leq l \leq k$ $\mathrm{nor}_n(a_l \cap s_n) > 0$, whence $\omega \setminus a_l \notin C_p$. It follows that $\mathcal{F}_p$ is a proper filter. Note that for any $s \in A_p$, $s \cap \mathrm{int}(p) \neq 0$, and so $\mathrm{int}(p) \in \mathcal{F}_p$.

Consider the forcing extension of $\mathbf{V}$ obtained by adding ω_1 Cohen reals. For each $\delta \leq \omega_1$, let $\mathbf{V}_\delta$ denote the extension by the first δ many of these. We assume that $\mathscr{A}$ remains MAD in $\mathbf{V}_{\omega_1}$.

For any family $\mathscr{B} \subset \mathcal{P}(\omega)$, $\mathcal{I}(\mathscr{B})$ is the ideal on ω generated by $\mathscr{B}$ together with the Fréchet ideal. For any ideal $\mathcal{I}$ on ω, $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$, and $\mathcal{I}^*$ is *the dual filter* to $\mathcal{I}$, that is $\mathcal{I}^* = \{\omega \setminus a : a \in \mathcal{I}\}$. For a filter $\mathcal{F}$ on ω, $\mathcal{F}^+ = (\mathcal{F}^*)^+$, where $\mathcal{F}^* = \{\omega \setminus a : a \in \mathcal{F}\}$ is *the dual ideal* to $\mathcal{F}$. For a family $\mathscr{B} \subset \mathcal{P}(\omega)$, we use $\mathcal{F}(\mathscr{B})$ to denote $(\mathcal{I}(\mathscr{B}))^*$. A filter $\mathcal{F}$ on ω is said to be P^+ if for any sequence $\langle b_n : n \in \omega\rangle$ with the property that $\forall n \in \omega\, [b_n \in \mathcal{F}^+ \wedge b_{n+1} \subset b_n]$, there exists $b \in \mathcal{F}^+$ such that $\forall n \in \omega\, [b \subset^* b_n]$.

Lemma 5. *In $\mathbf{V}_{\omega_1}$, let $\mathcal{F}$ be any F_σ filter and suppose that $\mathcal{G}$, the filter generated by $\mathcal{F} \cup \mathcal{F}(\mathscr{A})$, is a proper filter. Then $\mathcal{G}$ is P^+.*

Proof. Work in $\mathbf{V}_{\omega_1}$. Fix $\langle b_n : n \in \omega\rangle$ such that $b_{n+1} \subset b_n$ and each $b_n \in \mathcal{G}^+$. Write $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{T}_n$, where each $\mathcal{T}_n$ is a compact subset of $\mathcal{P}(\omega)$. Fix $\delta < \omega_1$ such that $\langle b_n : n \in \omega\rangle \in \mathbf{V}_\delta$ and (the code for) $\langle \mathcal{T}_n : n \in \omega\rangle \in \mathbf{V}_\delta$. In $\mathbf{V}_\delta$, observe that for any $\alpha_0, \ldots, \alpha_k < \omega_1$, any $n \in \omega$, any $(a_0, \ldots a_k) \in Y_{\alpha_0} \times \cdots \times Y_{\alpha_k}$, any $c \in \mathcal{T}_n$, and any $m \in \omega$, $b_m \cap c \cap (\omega \setminus a_0) \cap \cdots \cap (\omega \setminus a_k)$ is infinite. Therefore, by a standard compactness argument, for each $\alpha_0, \ldots, \alpha_k < \omega_1$, $n, m, l \in \omega$, there is a finite set $s \subset b_m \setminus l$ such that

$$(*) \quad \forall\, (a_0, \ldots a_k) \in Y_{\alpha_0} \times \cdots \times Y_{\alpha_k} \forall c \in \mathcal{T}_n\, [s \cap c \cap (\omega \setminus a_0) \cap \cdots \cap (\omega \setminus a_k) \neq 0]\,.$$

Note that $(*)$ is absolute between $\mathbf{V}_\delta$ and $\mathbf{V}_{\omega_1}$. Still in $\mathbf{V}_\delta$, consider the natural poset $\mathbb{P}$ for adding a pseudo-intersection to $\langle b_n : n \in \omega\rangle$ using finite conditions. $\mathbb{P}$ is forcing equivalent to Cohen forcing. So in $\mathbf{V}_{\omega_1}$, there is a set b which is $(\mathbf{V}_\delta, \mathbb{P})$

generic. Clearly, $\forall n \in \omega\, [b \subset^* b_n]$. Also, by genericity, for each $\alpha_0, \dots, \alpha_k < \omega_1$, $n, l \in \omega$, there is $s \subset b \setminus l$ such that $(*)$ holds. Thus $b \in \mathcal{G}^+$. ⊣

Definition 6. For an ultrafilter $\mathcal{U}$, a $\mathcal{U}$-tree is a tree $T \subset \omega^{<\omega}$ such that $\forall s \in T\, [\mathrm{succ}_T(s) \in \mathcal{U}]$ and $\forall f \in [T] \forall n \in \omega\, [f(n) < f(n+1)]$. Thus each $f \in [T]$ determines an element of $[\omega]^\omega$ in a natural way. We will often confuse these below.

Lemma 7. *In $\mathbf{V}_{\omega_1}$, suppose that $\mathcal{F}$ is a F_σ filter such that $\mathcal{G}$, the filter generated by $\mathcal{F} \cup \mathcal{F}(\mathscr{A})$, is proper. Suppose $b \in \mathcal{G}^+$. Then for each $\alpha_0, \dots, \alpha_k < \omega_1$, there is a $c \in [b]^\omega$ such that $c \in \mathcal{G}^+$ and $\forall (a_0, \dots, a_k) \in X_{\alpha_0} \times \dots \times X_{\alpha_k}\, [|(a_0 \cup \dots \cup a_k) \cap c| < \omega]$.*

Proof. Let $\mathcal{E}$ be the filter generated by $\mathcal{G} \cup \{b\}$, and let $\mathcal{I}$ be $\mathcal{E}^*$, the dual ideal. Consider the forcing with $\mathcal{P}(\omega)/\mathcal{I}$. By Lemma 5, this forcing does not add any reals and adds a P-point $\mathcal{U} \supset \mathcal{E}$. Work in $\mathbf{V}_{\omega_1}^{\mathcal{P}(\omega)/\mathcal{I}}$. Fix $0 \le i \le k$ and let $\mathcal{I}(X_{\alpha_i})$ be the ideal generated by X_{α_i}. This is analytic. By a theorem of Blass [Bl], there is a $\mathcal{U}$-tree T such that either $[T] \subset \mathcal{I}(X_{\alpha_i})$ or $[T] \cap \mathcal{I}(X_{\alpha_i}) = 0$. As $\mathcal{U}$ is a P-point, without loss of generality, there is a set $c_i \in [b]^\omega \cap \mathcal{U}$ such that $\forall s \in T\, [\mathrm{succ}_T(s) =^* c_i]$. We claim that $\forall a \in X_{\alpha_i}\, [|a \cap c_i| < \omega]$. Suppose not. Then it is possible to choose $f \in [T]$ such that $f \in \mathcal{I}(X_{\alpha_i})$. On the other hand, $c_i \in \mathcal{I}^+(\mathscr{A})$. As $\mathcal{P}(\omega)/\mathcal{I}$ adds no new reals, $\mathscr{A}$ is MAD in $\mathbf{V}_{\omega_1}^{\mathcal{P}(\omega)/\mathcal{I}}$, and so $\exists^\infty a \in \mathscr{A}\, [|a \cap c_i| = \omega]$. But then it is possible to choose $f \in [T]$ such that $\exists^\infty a \in \mathscr{A}\, [|a \cap f| = \omega]$, whence $f \notin \mathcal{I}(X_{\alpha_i})$. This contradicts the choice of T. Now, put $c = \bigcap_{0 \le i \le k} c_i$. $c \in [b]^\omega \cap \mathcal{U}$. Therefore, $c \in \mathcal{G}^+$. Also, it is clear that $\forall (a_0, \dots, a_k) \in X_{\alpha_0} \times \dots \times X_{\alpha_k}\, [|(a_0 \cup \dots \cup a_k) \cap c| < \omega]$. Since $\mathcal{P}(\omega)/\mathcal{I}$ did not add any reals, $c \in \mathbf{V}_{\omega_1}$, and we are done. ⊣

Definition 8. A 0-condition p is said to be a 1-*condition* if for each $a \in \mathcal{I}(\mathscr{A})$ and for each $k \in \omega$, there is $n \in \omega$ such that $\mathrm{nor}_n(s_n \setminus a) \ge k$.

The next lemma is the major new ingredient in the proof. Most of the extra work needed to deal with $\mathfrak{a}_{closed}$ rather than $\mathfrak{a}$ is contained in it.

Lemma 9. *Work in $\mathbf{V}_{\omega_1}$. Let p be a 0-condition and let $c \subset \omega$. Then the following are equivalent:*

1. *for every $\alpha_0, \dots, \alpha_k < \omega_1$, there exists a 1-condition q such that $q \le_0 p$, $\forall (a_0, \dots, a_k) \in X_{\alpha_0} \times \dots \times X_{\alpha_k}\, [|\mathrm{int}(q) \cap (a_0 \cup \dots \cup a_k)| < \omega]$, and $\mathrm{int}(q) \subset c$;*
2. *the filter generated by $\mathcal{F}_p \cup \mathcal{F}(\mathscr{A}) \cup \{c\}$ is proper.*

Proof. Assume (1), and suppose for a contradiction that there exist $b_0, \dots, b_l \in C_p$, $\alpha_0, \dots, \alpha_k < \omega_1$, $(a_0, \dots, a_k) \in X_{\alpha_0} \times \dots \times X_{\alpha_k}$, and $i \in \omega$ such that $c \cap \mathrm{int}(p) \cap b_0 \cap \dots \cap b_l \cap (\omega \setminus a_0) \cap \dots \cap (\omega \setminus a_k) \subset i$. Applying (1), find $q \le_0 p$ such that $\mathrm{int}(q) \subset c$, and $\mathrm{int}(q) \cap (a_0 \cup \dots \cup a_k)$ is finite. Find $n \in \omega$ such that $\mathrm{nor}^q_n(s^q_n) > l+1$, $i \cap s^q_n = 0$, and $(a_0 \cup \dots \cup a_k) \cap s^q_n = 0$. Since $s^q_n \subset \mathrm{int}(p) \cap c$, it follows that $s^q_n \subset (\omega \setminus b_0) \cup \dots \cup (\omega \setminus b_l)$. But then, for some $0 \le j \le l$, $\mathrm{nor}^q_n((\omega \setminus b_j) \cap s^q_n) > 0$. So there must be $m \in$ omega such that $\mathrm{nor}^p_m(s^p_m \cap (\omega \setminus b_j) \cap s^q_n) > 0$, whence $(\omega \setminus b_j) \cap s^q_n \in A_p$. This, however, means that $b_j \notin C_p$, a contradiction.

Next, suppose that $\mathcal{F}_p \cup \mathcal{F}(\mathscr{A}) \cup \{c\}$ generates a proper filter. We will prove (1). Let $\mathcal{G}$ denote the filter generated by $\mathcal{F}_p \cup \mathcal{F}(\mathscr{A}) \cup \{c\}$. First notice the following things about A_p. If $s \in A_p$, then $|s| > 1$. Next, if $s \subset t$, and $s \in A_p$, then $t \in A_p$. Finally, if $b \in \mathcal{G}^+$, then $\exists s \in A_p\, [s \subset b]$. Now, we define *the norm induced by* A_p, $\mathrm{nor} : [\omega]^{<\omega} \to \omega$ by the following clauses:

- $\mathrm{nor}(s) \geq 0$, for every $s \in [\omega]^{<\omega}$;
- $\mathrm{nor}(s) \geq 1$ iff $s \in A_p$;
- for $n > 1$, $\mathrm{nor}(s) \geq n$ iff for every s_0, s_1 such that $s = s_0 \cup s_1$, there is $i \in 2$ such that $\mathrm{nor}(s_i) \geq n-1$;
- $\mathrm{nor}(s) = \max\{n \in \omega : \mathrm{nor}(s) \geq n\}$.

It is easy to check that nor is well defined and is a norm on ω. Next, we check by induction on $n \in \omega$ that for any $b \in \mathcal{G}^+$, $\exists s \subset b\,[\mathrm{nor}(s) \geq n]$. If $n = 0$, then there is nothing to prove. For $n = 1$, use the previous observation that $\exists s \in A_p\,[s \subset b]$. Suppose that $n > 1$ and that the claim is true for $n-1$. Suppose for a contradiction that it fails for n. In particular, for every $k \in \omega$, $\mathrm{nor}(b \cap k) \ngeq n$, and so there exist b_0^k, b_1^k such that $b \cap k = b_0^k \cup b_1^k$, and neither b_0^k nor b_1^k contains a set s such that $\mathrm{nor}(s) \geq n-1$. By a standard König's Lemma argument, this gives us b_0, b_1 such that $b = b_0 \cup b_1$ and neither b_0 nor b_1 contains a set s with $\mathrm{nor}(s) \geq n-1$. However, either b_0 or b_1 is in $\mathcal{G}^+$, which contradicts the induction hypothesis.

Now, fix $\alpha_0, \dots, \alpha_k < \omega_1$. As $\mathcal{F}_p$ is a F_σ filter and as $\mathrm{int}(p) \cap c$ is positive for the filter generated by $\mathcal{F}_p \cup \mathcal{F}(\mathscr{A})$, Lemma 7 applies and implies that there is a set $d \in [\mathrm{int}(p) \cap c]^\omega$ which is positive for the filter generated by $\mathcal{F}_p \cup \mathcal{F}(\mathscr{A})$, and $\forall (a_0, \dots, a_k) \in X_{\alpha_0} \times \cdots \times X_{\alpha_k}\,[|(a_0 \cup \cdots \cup a_k) \cap d| < \omega]$. Of course, $d \in \mathcal{G}^+$. Therefore, for any $a \in \mathcal{I}(\mathscr{A})$, and for any $n \in \omega$, there is a $s \subset d$ such that $\mathrm{nor}(s) \geq n$ and $a \cap s = 0$. Choose $\delta < \omega_1$ such that p, c, d, and nor are in $\mathbf{V}_\delta$. Now, work in $\mathbf{V}_\delta$. Define a poset $\mathbb{P}$ as follows. For $s \in [d]^{<\omega} \setminus \{0\}$, let m_s denote $\min\{m \in \omega : s \cap s_m^p \neq 0\}$ and let m^s denote $\max\{m \in \omega : s\, caps_m^p \neq 0\}$. $\mathbb{P}$ consists of all $\sigma : \mathrm{dom}(\sigma) \to [d]^{<\omega}$ such that:

- $\mathrm{dom}(\sigma) \in \omega$ and for each $i < \mathrm{dom}(\sigma)$, $\mathrm{nor}(\sigma(i)) > 0$;
- for any $i < i+1 < \mathrm{dom}(\sigma)$, $\langle \sigma(i), \mathrm{nor} \restriction \sigma(i) \rangle < \langle \sigma(i+1), \mathrm{nor} \restriction \sigma(i+1) \rangle$ and also $m^{\sigma(i)} < m_{\sigma(i+1)}$.

For $\sigma, \tau \in \mathbb{P}$, $\tau \leq \sigma$ iff $\tau \supset \sigma$. Fix $\beta_0, \dots, \beta_l < \omega_1$, $n, m \in \omega$. For any $(a_0, \dots, a_l) \in Y_{\beta_0} \times \cdots \times Y_{\beta_l}$, there is a $s \subset d \setminus m$ such that $s \cap (a_0 \cup \cdots \cup a_l) = 0$ and $\mathrm{nor}(s) \geq n$. Again, by a compactness argument, there is a set $s \subset d \setminus m$ such that

$$(*) \qquad \forall (a_0, \dots, a_l) \in Y_{\beta_0} \times \cdots \times Y_{\beta_l} \exists t \subset s\,[(a_0 \cup \cdots \cup a_l) \cap t = 0 \wedge \mathrm{nor}(t) \geq n]\,.$$

Note that $(*)$ is absolute between $\mathbf{V}_\delta$ and $\mathbf{V}_{\omega_1}$. Now, for each $\beta_0, \dots, \beta_l < \omega_1$ and $n \in \omega$,

$$\{\tau \in \mathbb{P} : \exists i < \mathrm{dom}(\tau)\,[\tau(i) \text{ satisfies } (*) \text{ with respect to } \beta_0, \dots, \beta_l, n]\}$$

is dense in $\mathbb{P}$. Since $\mathbb{P}$ is forcing equivalent to Cohen forcing, there is a function $f : \omega \to [d]^{<\omega}$ in $\mathbf{V}_{\omega_1}$ which is $(\mathbf{V}_\delta, \mathbb{P})$-generic. For each $i \in \omega$, put $c_i^q = \langle f(i), \mathrm{nor} \restriction f(i) \rangle$. Put $q = \langle s^p, \langle c_i^q : i \in \omega \rangle \rangle$. It is clear that q is a 0-condition and that $q \leq_0 p$. It is also clear that $\mathrm{int}(q) \subset c$. By genericity of f, for each $\beta_0, \dots, \beta_l < \omega_1$ and $n \in \omega$, there is $i \in \omega$ such that s_i^q satisfies $(*)$ with respect to $\beta_0, \dots, \beta_l$ and n. It follows that q is a 1-condition, and we are done. ⊣

Corollary 10. *There are* 1*-conditions. Moreover, given any* 1*-condition p and $\alpha_0, \dots, \alpha_k < \omega_1$, there is a* 1*-condition $q \leq p$ such that $\forall (a_0, \dots, a_k) \in X_{\alpha_0} \times \cdots \times X_{\alpha_k}\,[|(a_0 \cup \cdots \cup a_k) \cap \mathrm{int}(q)| < \omega]$.*

Proof. For the second statement, note that if p is a 1-condition, then the filter generated by $\mathcal{F}_p \cup \mathcal{F}(\mathscr{A})$ is proper. Now, apply Lemma 9.

The first statement is a corollary of the proof of Lemma 9. For example, let $A = [\omega]^{\geq 2}$, and let nor, the norm induced by A, be defined as in the proof of Lemma 9. Let $\mathbb{P}$ be defined (with $d = \omega$) as in the proof of Lemma 9, leaving out any mention about $m^{\sigma(i)}$ and $m_{\sigma(i+1)}$, which are irrelevant here. Then an appropriate generic for $\mathbb{P}$ yields a 1-condition. $\dashv$

From this point on the argument is fairly standard, and follows Shelah [Sh2].

Definition 11. $\mathbb{P}_0 = \{p : p \text{ is a 0-condition}\}$. $\mathbb{P}_1 = \{p : p \text{ is a 1-condition}\}$. The ordering on both $\mathbb{P}_0$ and $\mathbb{P}_1$ is $\leq$.

Fix $p \in \mathbb{P}_0$. Suppose $t \in [\mathrm{int}(p)]^{<\omega}$. Define $m^p_t = \max\{m \in \omega : s^p_m \cap t \neq 0\}$, with the convention that $m^p_t = -1$ when $t = 0$. For $t \in [\mathrm{int}(p)]^{<\omega}$ and $n > m^p_t$, $p(t, n)$ is the 0-condition defined as follows. $s^{p(t,n)} = s^p \cup t$, and for all $i \in \omega$, $c^{p(t,n)}_i = c^p_{i+n}$. It is clear that $p(t, n) \leq p$.

The poset $\mathbb{P}_0$ is proper and does not add dominating reals. Consult either [Sh2] or [Ab] for a proof of this. We will work towards showing that $\mathbb{P}_1$ is proper. We first make some basic observations about the above definitions. Fix $p \in \mathbb{P}_0$ and suppose $q \leq_0 p$. Suppose $t \in [\mathrm{int}(q)]^{<\omega}$. Then $m^q_t \leq m^p_t$. Moreover, if $k > m^p_t$, then $q(t, k) \leq_0 p(t, k)$. Also, suppose that $p, q \in \mathbb{P}_0$ with $q \leq_0 p$. Suppose $t \in [\mathrm{int}(q)]^{<\omega}$ and suppose that $k > m^q_t$ and that $l > m^p_t$. If for each $m \geq k$, $s^q_m \subset \bigcup_{j\in[l,\infty)} s^p_j$, then $q(t, k) \leq_0 p(t, l)$. To avoid unnecessary repetitions, all conditions belong to $\mathbb{P}_1$ from this point on unless specified. Also, unless specified, we are working inside $\mathbf{V}_{\omega_1}$.

Lemma 12. *Let $\mathring{x} \in \mathbf{V}^{\mathbb{P}_1}_{\omega_1}$ such that $\Vdash_1 \mathring{x} \in \mathbf{V}_{\omega_1}$. Fix p, $k \in \omega \setminus \{0\}$, and $t \subset \bigcup_{m\in[0,k)} s^p_m$. Then there is $\bar{p} \leq_k p$ such that for any $q \leq_k \bar{p}$, if there exists $r \leq q$ such that $s^r \setminus s^p = t$ and $r \Vdash_1 \mathring{x} = x$, then $q(t, k) \Vdash_1 \mathring{x} = x$.*

Proof. $\bar{p}$ is gotten as follows. First suppose that there is a $\bar{q} \leq_0 p(t, k)$ and $x \in \mathbf{V}_{\omega_1}$ such that $\bar{q} \Vdash_1 \mathring{x} = x$. We may assume that $\mathrm{nor}^{\bar{q}}_0 \left(s^{\bar{q}}_0\right) > \mathrm{nor}^p_{k-1} \left(s^p_{k-1}\right)$. Now define $\bar{p}$, by $s^{\bar{p}} = s^p$, $c^{\bar{p}}_m = c^p_m$, for $m < k$, and $c^{\bar{p}}_m = c^{\bar{q}}_{m-k}$, for $m \geq k$. If there is no such $\bar{q}$, then simply set $\bar{p} = p$. In either case, it is clear that $\bar{p} \leq_k p$.

Now, fix $q \leq_k \bar{p}$. Note that if the first case happens above, then $q(t, k) \leq_0 \bar{q}$, and so $q(t, k) \Vdash_1 \mathring{x} = x$. Suppose $r \leq q$ such that $s^r \setminus s^p = t$ and $y \in \mathbf{V}_{\omega_1}$ such that $r \Vdash_1 \mathring{x} = y$. First, we claim that the first case must have happened above. Suppose not. Then $\bar{p} = p$. We may assume that $s^r_0 \subset \bigcup_{m\in[k,\infty)} s^q_m$. But then $r \leq_0 p(t, k)$, which contradicts the supposition that the first case did not occur. So the first case occurs, and therefore, $q(t, k) \Vdash_1 \mathring{x} = x$. Again, we may assume that $s^r_0 \subset \bigcup_{m\in[k,\infty)} s^q_m$. But then $r \leq_0 q(t, k)$, whence $x = y$. $\dashv$

Lemma 13. *Let $\mathring{x} \in \mathbf{V}^{\mathbb{P}_1}_{\omega_1}$ such that $\Vdash_1 \mathring{x} \in \mathbf{V}_{\omega_1}$. Fix p, $k \in \omega \setminus \{0\}$. There exists $\bar{p} \leq_k p$ such that*

($\dagger_1$) *for any $q \leq_k \bar{p}$ and for any $t \subset \bigcup_{m\in[0,k)} s^p_m$, if there exists $r \leq q$ and $x \in \mathbf{V}_{\omega_1}$ such that $s^r \setminus s^p = t$ and $r \Vdash_1 \mathring{x} = x$, then $q(t, k) \Vdash_1 \mathring{x} = x$.*

Proof. Let $t_0, \dots, t_l$ enumerate all $t \subset \bigcup_{m\in[0,k)} s^p_m$. Now construct a sequence $p = p_{-1}\ {}_k\!\geq p_0\ {}_k\!\geq \cdots\ {}_k\!\geq p_l = \bar{p}$ as follows. For $-1 \leq i < l$, suppose $p_i \leq_k p$ is given. Note that $t_{i+1} \subset \bigcup_{m\in[0,k)} s^{p_i}_m$. So apply Lemma 12 to find $p_{i+1} \leq_k p_i$ such that for any $q \leq_k p_{i+1}$, if there are $r \leq q$ and $x \in \mathbf{V}_{\omega_1}$ such that $s^r \setminus s^p = t_{i+1}$ and $r \Vdash_1 \mathring{x} = x$, then $q(t_{i+1}, k) \Vdash_1 \mathring{x} = x$. It is clear that $\bar{p}$ is as needed. $\dashv$

Lemma 14. *Fix $p \in \mathbb{P}_1$ and $\mathring{f} \in \mathbf{V}_{\omega_1}^{\mathbb{P}_1}$ such that $\Vdash_1 \mathring{f} \in {}^{\omega}(\mathbf{V}_{\omega_1})$. Then there is a $\bar{p} \leq_0 p$ such that*

$(\dagger_2)$ *for any $q \leq_0 \bar{p}$, for any $t \in [\mathrm{int}(q)]^{<\omega}$, and for any $i \in \omega$, there is a $k > m^q_t$ such that if there is a $r \leq q$ and $x \in \mathbf{V}_{\omega_1}$ such that $s^r \setminus s^p = t$ and $r \Vdash_1 \mathring{f}(i) = x$, then $q(t,k) \Vdash_1 \mathring{f}(i) = x$.*

Proof. Define functions $\Sigma : \omega^{<\omega} \to \mathbb{P}_1$ and $\Delta : \omega^{<\omega} \setminus \{0\} \to \omega \setminus \{0\}$ with the following properties:

(1) $\Sigma(0) = p$ and for each $\sigma \in \omega^{<\omega}$ and $j \in \omega$, $\Sigma(\sigma^\frown\langle j\rangle) \leq_{\Delta(\sigma^\frown\langle j\rangle)} \Sigma(\sigma)$;
(2) for each $\sigma \in \omega^{<\omega} \setminus \{0\}$, and for each $j \in \omega$, $\Delta(\sigma^\frown\langle j\rangle) > \Delta(\sigma)$; also, for each $\sigma \in \omega^{<\omega}$ and $k \in \omega$, there is a $j \in \omega$ such that $\Delta(\sigma^\frown\langle j\rangle) > k$;
(3) for each $\sigma \in \omega^{<\omega}$, $j \in \omega$, $i < \Delta(\sigma^\frown\langle j\rangle)$, $(\dagger_1)$ holds with $\bar{p}$ as $\Sigma(\sigma^\frown\langle j\rangle)$, k as $\Delta(\sigma^\frown\langle j\rangle)$, p as $\Sigma(\sigma)$, and $\mathring{x}$ as $\mathring{f}(i)$.

By Lemma 13 it is possible to define such functions Σ and Δ. Now, fix $g \in \omega^\omega$. The hypotheses of Lemma 3 are satisfied when p_n is taken to be $\Sigma(g \restriction n)$ and k_n as $\Delta(g \restriction n{+}1)$. Let q_g be the 0-condition defined as in Lemma 3. By Lemma 3, for each $n \in \omega$, $q_g \leq_{\Delta(g\restriction n+1)} \Sigma(g \restriction n)$. Suppose for a moment that there is $g \in \omega^\omega$ such that $q_g \in \mathbb{P}_1$. We first check that setting $q_g = \bar{p}$ does the job. Suppose $q \leq_0 q_g$. Fix $t \in [\mathrm{int}(q)]^{<\omega}$ and $i \in \omega$. Find $n \in \omega$ such that $\Delta(g \restriction n+1) > \max\left\{m^{q_g}_t, i\right\}$. Observe that $m^q_t < \Delta(g \restriction n+1)$. Thus $t \subset \bigcup_{m\in[0,\Delta(g\restriction n+1))} s^{q^g}_m$. As $q_g \leq_{\Delta(g\restriction n+1)} \Sigma(g \restriction n)$, $t \subset \bigcup_{m\in[0,\Delta(g\restriction n+1))} s^{\Sigma(g\restriction n)}_m$. We know that $(\dagger_1)$ holds with $\bar{p}$ as $\Sigma(g \restriction n+1)$, k as $\Delta(g \restriction n+1)$, p as $\Sigma(g \restriction n)$, and $\mathring{x}$ as $\mathring{f}(i)$. Note that $q_g \leq_{\Delta(g\restriction n+2)} \Sigma(g \restriction n+1)$, and so $q_g \leq_{\Delta(g\restriction n+1)} \Sigma(g \restriction n+1)$. Now, suppose there exists $r \leq q$ and $x \in \mathbf{V}_{\omega_1}$ such that $s^r \setminus s^p = t$ and $r \Vdash_1 \mathring{f}(i) = x$. Note that $s^p = s^{\Sigma(g\restriction n)}$ and that $q \leq q_g$. Therefore, $r \leq q_g$ and $s^r \setminus s^{\Sigma(g\restriction n)} = t$. Applying $(\dagger_1)$, we conclude that $q_g(t, \Delta(g \restriction n+1)) \Vdash_1 \mathring{f}(i) = x$. But since $q(t, \Delta(g \restriction n+1)) \leq_0 q_g(t, \Delta(g \restriction n+1))$, $q(t, \Delta(g \restriction n+1)) \Vdash_1 \mathring{f}(i) = x$, and we are done.

Therefore, it is enough to find $g \in \omega^\omega$ such that $q_g \in \mathbb{P}_1$. Find $\delta < \omega_1$ such that $\Sigma, \Delta \in \mathbf{V}_\delta$. Work in $\mathbf{V}_\delta$. View $\omega^{<\omega}$ as a forcing poset with $\tau \leq \sigma$ iff $\tau \supset \sigma$. Fix $\sigma \in \omega^{<\omega}$, $\alpha_0, \dots, \alpha_k < \omega_1$, and $n, m \in \omega$. Then for each $(a_0, \dots, a_k) \in Y_{\alpha_0} \times \cdots \times Y_{\alpha_k}$, there is $i \in \omega$ and $t \subset s^{\Sigma(\sigma)}_i$ with $\mathrm{nor}^{\Sigma(\sigma)}_i(t) \geq n$ such that $t \cap (m \cup a_0 \cup \cdots \cup a_k) = 0$. Again, by a compactness argument, there exists $j \in \omega$ such that

$$(*) \qquad \forall (a_0, \dots, a_k) \in Y_{\alpha_0} \times \cdots \times Y_{\alpha_k} \exists i \leq j \; \exists t \subset s^{\Sigma(\sigma)}_i \left[\mathrm{nor}^{\Sigma(\sigma)}_i(t) \geq n \wedge t \cap (m \cup a_0 \cup \cdots \cup a_k) = 0\right].$$

Note that $(*)$ is absolute between $\mathbf{V}_\delta$ and $\mathbf{V}_{\omega_1}$. It follows that for any $\alpha_0, \dots, \alpha_k < \omega_1$ and $n, m \in \omega$, the set

$$\{\tau \in \omega^{<\omega} \setminus \{0\} : \Delta(\tau) - 1 \text{ satisfies } (*) \text{ with respect to } \tau \restriction |\tau| - 1, \alpha_0, \dots, \alpha_k, n, m\}$$

is dense in $\omega^{<\omega}$. There is a $g \in \mathbf{V}_{\omega_1}$ which is $(\mathbf{V}_\delta, \omega^{<\omega})$-generic. By genericity, for each $\alpha_0, \dots, \alpha_k < \omega_1$, and $n, m \in \omega$, there is a $l \in \omega$ such that $\Delta(g \restriction l+1) - 1$ satisfies $(*)$ with respect to $g \restriction l$, $\alpha_0, \dots, \alpha_k, n, m$. Since $q_g \leq_{\Delta(g\restriction l+1)} \Sigma(g \restriction l)$, it follows that $q_g \in \mathbb{P}_1$. $\dashv$

An easy corollary of Lemma 14 is the properness of $\mathbb{P}_1$. The details are left to the reader.

Corollary 15. $\mathbb{P}_1$ *is proper.*

We next work towards showing that if $\mathbb{P}_0$ does not destroy $\mathscr{A}$, then $\mathbb{P}_1$ does not add dominating reals, and more.

Definition 16. Fix $\mathring{f} \in \mathbf{V}_{\omega_1}^{\mathbb{P}_1}$ such that $\Vdash_1 \mathring{f} \in {}^{\omega}(\mathbf{V}_{\omega_1})$. Let $p \in \mathbb{P}_1$ satisfy $(\dagger_2)$ of Lemma 14 with respect to $\mathring{f}$. For each $i \in \omega$, define

$$B(p, \mathring{f}, i) = \left\{t \in [\mathrm{int}(p)]^{<\omega} : \exists k > m_t^p \exists x \in \mathbf{V}_{\omega_1} \left[p(t,k) \Vdash_1 \mathring{f}(i) = x\right]\right\}.$$

Note that if $\mathring{f}$ and p are as in Definition 16, and if $q \leq_0 p$, then q also satisfies $(\dagger_2)$ with respect to $\mathring{f}$ and that $B(q, \mathring{f}, i) = [\mathrm{int}(q)]^{<\omega} \cap B(p, \mathring{f}, i)$, for each $i \in \omega$.

Lemma 17. *Let $\mathring{f}$ and p be as in Definition 16. Fix $k \in \omega \setminus \{0\}$. There exists $\bar{p} \leq_k p$ such that*

$$(\dagger_3) \quad \forall t \subset \bigcup_{m \in [0,k)} s_m^p \forall i < k \forall m \geq k$$
$$\forall u \subset s_m^{\bar{p}} \left[\mathrm{nor}_m^{\bar{p}}(u) > 0 \implies \exists v \subset u \left[t \cup v \in B(\bar{p}, \mathring{f}, i)\right]\right].$$

Proof. Let A be the set of all $u \in \left[\bigcup_{n \in [k,\infty)} s_n^p\right]^{<\omega}$ such that:

(1) for some $m \in \omega$, $\mathrm{nor}_m^p(s_m^p \cap u) > 0$;
(2) for each $t \subset \bigcup_{m \in [0,k)} s_m^p$ and $i < k$, there exists $v \subset u$ such that $t \cup v \in B(p, \mathring{f}, i)$.

It is easy to see that for any $u \in A$, $|u| > 1$ and that if $u \subset w$, then $w \in A$. Let $\mathcal{G}$ denote the filter generated by $\mathcal{F}_p \cup \mathcal{F}(\mathscr{A})$. Note that $\mathcal{G}$ is a proper filter. Fix $c \in \mathcal{G}^+$. Then the filter generated by $\mathcal{G} \cup \{c\}$ is proper, and so by Lemma 9, there is a 1-condition $q \leq_0 p$ such that $\mathrm{int}(q) \subset c$. Let n_0 be least such that for each $n \geq n_0$, $s_n^q \subset \bigcup_{m \in [k,\infty)} s_m^p$, and $\mathrm{nor}_n^q(s_n^q) > \mathrm{nor}_{k-1}^p(s_{k-1}^p)$. Define $\bar{q}$ such that $s^{\bar{q}} = s^q$, for each $m \in [0,k)$, $s_m^{\bar{q}} = s_m^p$, and for all $m \in [k,\infty)$, $s_m^{\bar{q}} = s_{(m-k)+n_0}^q$. It is clear that $\bar{q}$ is a 1-condition and that $\bar{q} \leq_k p$. Now, fix $t \subset \bigcup_{m \in [0,k)} s_m^p$ and $i < k$. Find $r \leq \bar{q}(t,k)$ and $x \in \mathbf{V}_{\omega_1}$ such that $r \Vdash_1 \mathring{f}(i) = x$. Let $v = s^r \setminus (s^p \cup t)$ and note that since p satisfies $(\dagger_2)$, $t \cup v \in B(p, \mathring{f}, i)$. Find $n(t,i) > n_0$ such that $v \subset \bigcup_{m \in [n_0, n(t,i))} s_m^q$. Put $n = \max\left\{n(t,i) : t \subset \bigcup_{m \in [0,k)} s_m^p \wedge i < k\right\}$. Let $u = \bigcup_{m \in [n_0,n)} s_m^q$. Observe that $u \in \left[\bigcup_{m \in [k,\infty)} s_m^p\right]^{<\omega}$. Since $s_{n_0}^q \subset u$, (1) is satisfied. Also by the way n is chosen, (2) is satisfied. Therefore $u \in A$. Since $u \subset \mathrm{int}(q) \subset c$, we conclude that for any $c \in \mathcal{G}^+$, there is a $u \in A$ such that $u \subset c$.

Now, let $\mathrm{nor} : [\omega]^{<\omega} \to \omega$ be the norm induced by A, defined exactly as in the proof of Lemma 9. Arguing as in Lemma 9, it is easy to prove that for any $c \in \mathcal{G}^+$ and $n \in \omega$, there is a $s \subset c$ with $\mathrm{nor}(s) \geq n$. Find a $\delta < \omega_1$ such that p and nor are in $\mathbf{V}_\delta$. Working in $\mathbf{V}_\delta$, define a poset $\mathbb{P}$ as follows. For a non-empty set $u \in [\mathrm{int}(p)]^{<\omega}$, m^u and m_u are defined as in the proof of Lemma 9. A condition in $\mathbb{P}$ is a function $\sigma : \mathrm{dom}(\sigma) \to [\mathrm{int}(p)]^{<\omega}$ such that:

(3) $\mathrm{dom}(\sigma) \in \omega$ and for each $i < \mathrm{dom}(\sigma)$, $\sigma(i) \subset \bigcup_{m \in [k,\infty)} s_m^p$ and $\mathrm{nor}(\sigma(i)) > \mathrm{nor}_{k-1}^p(s_{k-1}^p)$;

(4) for each $i < i+1 < \mathrm{dom}(\sigma)$, $\langle \sigma(i), \mathrm{nor} \restriction \sigma(i) \rangle < \langle \sigma(i+1), \mathrm{nor} \restriction \sigma(i+1) \rangle$, and $m^{\sigma(i)} < m_{\sigma(i+1)}$.

For $\sigma, \tau \in \mathbb{P}$, $\tau \le \sigma$ if $\tau \supset \sigma$. Given $\alpha_0, \dots, \alpha_l < \omega_1$, $m, n \in \omega$, and $(a_0, \dots, a_l) \in Y_{\alpha_0} \times \dots \times Y_{\alpha_l}$, there is a finite $u \subset \mathrm{int}(p) \setminus m$ such that $u \cap (a_0 \cup \dots \cup a_l) = 0$ and $\mathrm{nor}(u) \ge n$. So by a compactness argument, for each $\alpha_0, \dots, \alpha_l < \omega_1$, and $m, n \in \omega$, there is a finite $s \subset \mathrm{int}(p) \setminus m$ such that

$$(*) \quad \forall (a_0, \dots, a_l) \in Y_{\alpha_0} \times \dots \times Y_{\alpha_l} \exists u \subset s \, [u \cap (a_0 \cup \dots \cup a_l) = 0 \wedge \mathrm{nor}(u) \ge n] .$$

Observe that $(*)$ is absolute between $\mathbf{V}_\delta$ and $\mathbf{V}_{\omega_1}$. For each $\alpha_0, \dots, \alpha_l < \omega_1$ and $n \in \omega$, the set

$$\{\tau \in \mathbb{P} : \exists i < \mathrm{dom}(\tau) \, [\tau(i) \text{ satisfies } (*) \text{ with respect to } \alpha_0, \dots, \alpha_l, n]\}$$

is dense in $\mathbb{P}$. In $\mathbf{V}_{\omega_1}$, choose $f : \omega \to \left[\bigcup_{m \in [k,\infty)} s^p_m\right]^{<\omega}$ which is $(\mathbf{V}_\delta, \mathbb{P})$-generic. Define $\bar{p}$ as follows. $s^{\bar{p}} = s^p$. For each $m \in [0, k)$, $c^{\bar{p}}_m = c^p_m$. For $m \in [k, \infty)$, $c^{\bar{p}}_m = \langle f(m-k), \mathrm{nor} \restriction f(m-k) \rangle$. From the genericity of f, it follows that $\bar{p}$ is a 1-condition. Also, it is clear that $\bar{p} \le_k p$. Now, suppose that $t \subset \bigcup_{m \in [0,k)} s^p_m$ and $i < k$. Fix $m \ge k$ and $u \subset s^{\bar{p}}_m$ with $\mathrm{nor}(u) > 0$. Then $u \in A$, and so there is a $v \subset u$ such that $t \cup v \in B(p, \mathring{f}, i)$. As $B(\bar{p}, \mathring{f}, i) = [\mathrm{int}(\bar{p})]^{<\omega} \cap B(p, \mathring{f}, i)$, it follows that $t \cup v \in B(\bar{p}, \mathring{f}, i)$. ⊣

Note that if $\bar{p}$ satisfies $(\dagger_2)$ with respect to $\mathring{f}$ and it satisfies $(\dagger_3)$ with respect to $\mathring{f}$ and k, then any $q \le_k \bar{p}$ also satisfies $(\dagger_3)$ with respect to $\mathring{f}$ and k.

Lemma 18. *Let p and $\mathring{f}$ be as in Definition 16. There is a $\bar{p} \le_0 p$ such that*

$(\dagger_4)$ *for any $i \in \omega$, there is $k > i$ such that for any*

$t \subset \bigcup_{m \in [0,k)} s^{\bar{p}}_m, j < k, m \ge k$, and $u \subset s^{\bar{p}}_m$, if $\mathrm{nor}^{\bar{p}}_m(u) > 0$,

then there exists $v \subset u$ such that $t \cup v \in B(\bar{p}, \mathring{f}, j)$.

Proof. Define two functions $\Sigma : \omega^{<\omega} \to \mathbb{P}_1$ and $\Delta : \omega^{<\omega} \setminus \{0\} \to \omega \setminus \{0\}$ with the following properties:

(1) $\Sigma(0) = p$ and for each $\sigma \in \omega^{<\omega}$ and $j \in \omega$, $\Sigma(\sigma^\frown\langle j \rangle) \le_{\Delta(\sigma^\frown\langle j \rangle)} \Sigma(\sigma)$;
(2) for each $\sigma \in \omega^{<\omega} \setminus \{0\}$, and for each $j \in \omega$, $\Delta(\sigma^\frown\langle j \rangle) > \Delta(\sigma)$; also, for each $\sigma \in \omega^{<\omega}$ and $k \in \omega$, there is a $j \in \omega$ such that $\Delta(\sigma^\frown\langle j \rangle) > k$;
(3) for each $\sigma \in \omega^{<\omega}$ and $j \in \omega$, $\Sigma(\sigma^\frown\langle j \rangle)$ satisfies $(\dagger_3)$ with respect to $\mathring{f}$ and $\Delta(\sigma^\frown\langle j \rangle)$.

By Lemma 17 it is possible to find Σ and Δ with these properties. Note that for any $\sigma \in \omega^{<\omega}$, $\Sigma(\sigma) \le_0 p$. Therefore, $\Sigma(\sigma)$ satisfies $(\dagger_2)$ with respect to $\mathring{f}$. So Lemma 17 does apply to each $\Sigma(\sigma)$.

For each $g \in \omega^\omega$, let q_g be defined exactly as in the proof of Lemma 14. By the same argument as in Lemma 14, there exists $g \in \omega^\omega$ such that $q_g \in \mathbb{P}_1$. We argue that putting $\bar{p} = q_g$ works. Fix $i \in \omega$. Find $n \in \omega$ such that $\Delta(g \restriction n+1) > i$. Recall that $q_g \le_{\Delta(g \restriction n+1)} \Sigma(g \restriction n)$. Moreover, $q_g \le_{\Delta(g \restriction n+2)} \Sigma(g \restriction n+1)$, and so $q_g \le_{\Delta(g \restriction n+1)} \Sigma(g \restriction n+1)$. By (3), $\Sigma(g \restriction n+1)$ satisfies $(\dagger_3)$ with respect to $\mathring{f}$ and $\Delta(g \restriction n+1)$. Also, $\Sigma(g \restriction n+1)$ satisfies $(\dagger_2)$ with respect to $\mathring{f}$. It follows that q_g satisfies $(\dagger_3)$ with respect to $\mathring{f}$ and $\Delta(g \restriction n+1)$, and we are done. ⊣

Lemma 19. *Assume that* $\Vdash_0 \mathscr{A}$ *is MAD. Let* $p \in \mathbb{P}_1$. *There exists* $\{a_n : n \in \omega\} \subset \mathscr{A}$ *and* $\{q_n : n \in \omega\} \subset \mathbb{P}_0$ *such that:*

(1) $\forall n < n^* \, [a_n \neq a_{n^*}]$;
(2) $\forall n \in \omega \, [q_n \leq_0 p \wedge \operatorname{int}(q_n) \subset a_n]$.

Proof. Let $\mathring{x}$ be the canonical $\mathbb{P}_0$-name for the generic subset of ω added by $\mathbb{P}_0$. Fix $n \in \omega$ and suppose that $\{a_i : i < n\} \subset \mathscr{A}$ and $\{q_i : i < n\} \subset \mathbb{P}_0$ are given. We will show how to get a_n and q_n. Put $a = \bigcup_{i<n} a_i$. Then $a \in \mathcal{I}(\mathscr{A})$. Put $c = \operatorname{int}(p) \setminus a$. As p is a 1-condition, the filter generated by $\mathcal{F}_p \cup \mathcal{F}(\mathscr{A}) \cup \{c\}$ is proper. Apply Lemma 9 to find a 1-condition $\bar{p} \leq_0 p$ with $\operatorname{int}(\bar{p}) \subset c$. Since $\Vdash_0 \mathscr{A}$ is MAD, there is a 0-condition $q \leq \bar{p}$ and $\alpha < \omega_1$ such that $q \Vdash_0 \exists a^* \in X_\alpha \, [|a^* \cap \mathring{x}| = \omega]$. Note that for any $r \in \mathbb{P}_0$, $r \Vdash_0 \mathring{x} \subset^* \operatorname{int}(r)$. It follows that there can be no $r \in \mathbb{P}_0$ with $r \leq_0 q$ such that $\forall a^* \in X_\alpha \, [|\operatorname{int}(r) \cap a^*| < \omega]$. By Lemma 9, this means that the filter generated by $\mathcal{F}_q \cup \mathcal{F}(\mathscr{A})$ is not proper. Fix $b_0, \dots, b_l \in C_q$, $a_0^*, \dots a_k^* \in \mathscr{A}$, and $i \in \omega$ such that $b_0 \cap \dots \cap b_l \cap (\omega \setminus a_0^*) \cap \dots \cap (\omega \setminus a_k^*) \cap \operatorname{int}(q) \subset i$. Fix $m_0 \in \omega$ such that for all $m \geq m_0$, $s_m^q \cap i = 0$, and $\operatorname{nor}_m^q(s_m^q) > \max\{l, k\} + 1$. As $b_j \in C_q$ for any $0 \leq j \leq l$, it follows that for any $m \geq m_0$ there is a j_m with $0 \leq j_m \leq k$ such that $\operatorname{nor}_m^q \left(a_{j_m}^* \cap s_m^q\right) \geq (m - m_0) + 1$. So there is an infinite $X \subset [m_0, \infty)$ and $0 \leq j \leq k$ such that for each $m \in X$, $j_m = j$. Put $a_n = a_j^*$. Note that $a_n \cap \operatorname{int}(\bar{p}) \neq 0$, and so $a_n \neq a_i$ for any $i < n$. Define q_n as follows. $s^{q_n} = s^{\bar{p}} = s^p$. Choose $l_0 < l_1 < \cdots$, with $l_i \in X$ such that $\operatorname{nor}_{l_i}^q \left(a_n \cap s_{l_i}^q\right) < \operatorname{nor}_{l_{i+1}}^q \left(a_n \cap s_{l_{i+1}}^q\right)$. For each $i \in \omega$, define $c_i^{q_n} = \langle a_n \cap s_{l_i}^q, \operatorname{nor}_{l_i}^q \restriction \left(a_n \cap s_{l_i}^q\right)\rangle$. As $q \leq \bar{p}$, it is clear that $q_n \leq_0 \bar{p} \leq_0 p$. Also, $\operatorname{int}(q_n) \subset a_n$, and so q_n and a_n are as needed. $\dashv$

Lemma 20. *Assume that* $\Vdash_0 \mathscr{A}$ *is MAD. Let* $\mathring{f}$ *be as in Definition 16. Suppose that* $p \in \mathbb{P}_1$ *satisfies both* $(\dagger_2)$ *and* $(\dagger_4)$ *with respect to* $\mathring{f}$. *There exists a* 1*-condition* $q \leq_0 p$ *and* $\{a_n : n \in \omega\} \subset \mathscr{A}$ *with the following properties:*

(1) for all $n < n^*$, $a_n \neq a_{n^*}$*;*
(2) for each $n, l \in \omega$, $\forall^\infty m \in \omega \exists t \subset s_m^q \, [\operatorname{nor}_m^q(t) \geq l \wedge t \subset a_n]$*;*
(3) for any $k \in \omega$, $t \subset \bigcup_{m \in [0,k)} s_m^q$, *and* $u \subset s_k^q$, *if* $\operatorname{nor}_k^q(u) > 0$, *then there exists* $v \subset u$ *and* $x \in \mathbf{V}_{\omega_1}$ *such that* $q(t \cup v, k+1) \Vdash_1 \mathring{f}(k) = x$.

Proof. First apply Lemma 19 to p to find $\{a_n : n \in \omega\}$ and $\{q_n : n \in \omega\} \subset \mathbb{P}_0$ satisfying (1) and (2) of Lemma 19. Define $A = \{s \in [\omega]^{<\omega} : \exists n \in \omega \exists m \in \omega \, [\operatorname{nor}_m^{q_n}(s \cap s_m^{q_n}) > 0]\}$. Note that for any $s \in A$, $|s| > 1$ and that if $s \subset t$, then $t \in A$. Moreover, for any $s \in A$, there is $m \in \omega$ such that $\operatorname{nor}_m^p(s \cap s_m^p) > 0$. Let $\operatorname{nor} : [\omega]^{<\omega} \to \omega$ be the norm on ω induced by A, defined as in the proof of Lemma 9. Note that for any $n, m \in \omega$ and $s \subset s_m^{q_n}$, $\operatorname{nor}(s) \geq \operatorname{nor}_m^{q_n}(s)$. Next, recalling that p satisfies $(\dagger_2)$ with respect to $\mathring{f}$, for each $i \in \omega$ and $t \in B(p, \mathring{f}, i)$, fix $k_t^i > m_t^p$ such that $\exists x \in \mathbf{V}_{\omega_1} \left[p\left(t, k_t^i\right) \Vdash_1 \mathring{f}(i) = x\right]$.

Now, to get q proceed as follows. $s^q = s^p$. For each $i \in \omega$ choose $m_0, \dots, m_i \in \omega$ such that putting $s_i^q = s_{m_0}^{q_0} \cup \dots \cup s_{m_i}^{q_i}$, the following properties hold:

(4) for each $i < i+1$, $\max(s_i^q) < \min(s_{i+1}^q)$, $\max\{n \in \omega : s_n^p \cap s_i^q \neq 0\} < \min\{n \in \omega : s_n^p \cap s_{i+1}^q \neq 0\}$, and $\operatorname{nor}(s_i^q) < \operatorname{nor}(s_{i+1}^q)$ (recall that for all $j \in \omega$, $q_j \leq_0 p$; therefore, for any fixed $i \in \omega$, $s_{m_j}^{q_j} \subset \operatorname{int}(p)$, for each $0 \leq j \leq i$; also, $s_{m_0}^{q_0} \subset s_i^q$; therefore, s_i^q is a non-empty finite subset of $\operatorname{int}(p)$);

(5) for each $i \in \omega$, for each $t \subset \bigcup_{m \in [0,i)} s^q_m$, for each $u \subset s^q_i$, if $\mathrm{nor}(u) > 0$, then there exists $v \subset u$ such that $t \cup v \in B(p, \mathring{f}, i)$;

(6) for each $i \in \omega$, each $t \subset \bigcup_{m \in [0,i)} s^q_m$, and each $v \subset s^q_i$ such that $t \cup v \in B(p, \mathring{f}, i)$, $\forall m \geq i+1 \left[s^q_m \subset \bigcup_{n \in \left[k^i_{(t \cup v)}, \infty\right)} s^p_n \right]$;

(7) for each $i \in \omega$ and $0 \leq j \leq i$, $\mathrm{nor}^{q_j}_{m_j}\left(s^{q_j}_{m_j}\right) \geq i$.

Before showing how to do this for each $i \in \omega$, let us argue that it is enough to do so. First note that for any $j \in \omega$, $q_j \leq_0 p$, and so $s^{q_j} = s^p$. So since for any $i \in \omega$ and $l \in s^q_i$, there is some $0 \leq j \leq i$ such that $l \in s^{q_j}_{m_j}$, it follows that for all $l^* \in s^q$, $l^* < l$. Next, for any $i \in \omega$, $s^{q_0}_{m_0} \subset s^q_i$. So $0 < \mathrm{nor}^{q_0}_{m_0}\left(s^{q_0}_{m_0}\right) \leq \mathrm{nor}\left(s^{q_0}_{m_0}\right) \leq \mathrm{nor}(s^q_i)$. Therefore, if we put $c^q_i = \langle s^q_i, \mathrm{nor} \restriction s^q_i \rangle$, then $q = \langle s^q, \langle c^q_i : i \in \omega \rangle\rangle$ is a 0-condition, and $q \leq_0 p$. To check (2), fix $n, l \in \omega$. Suppose that $m \geq \max\{n, l\}$. Then there exists $m_n \in \omega$ such that $s^{q_n}_{m_n} \subset s^q_m$, and $\mathrm{nor}^{q_n}_{m_n}\left(s^{q_n}_{m_n}\right) \geq m \geq l$. However, $s^{q_n}_{m_n} \subset a_n$ and $\mathrm{nor}\left(s^{q_n}_{m_n}\right) \geq \mathrm{nor}^{q_n}_{m_n}\left(s^{q_n}_{m_n}\right) \geq l$. This verifies (2).

Using (2), it is easy to check that q is a 1-condition. With the next lemma in mind, we will verify a slightly stronger statement. Fix $X \in [\omega]^\omega$. Define $q_X = \langle s^q, \langle c^q_i : i \in X \rangle\rangle$. It is clear that q_X is a 0-condition and that $q_X \leq q$. We check that it is a 1-condition. Fix $a \in \mathcal{I}(\mathscr{A})$ and $l \in \omega$. Fix $n, k \in \omega$ such that $a \cap a_n \subset k$. Choose $m \in X$ such that $s^q_m \cap k = 0$ and there exists $t \subset s^q_m$ such that $t \subset a_n$ and $\mathrm{nor}^q_m(t) \geq l$. It is clear that $t \cap a = 0$, and this checks that q_X is a 1-condition.

For (3), fix $i \in \omega$, $t \subset \bigcup_{m \in [0,i)} s^q_m$, and $u \subset s^q_i$ such that $\mathrm{nor}^q_i(u) > 0$. By (5), there is a $v \subset u$ such that $t \cup v \in B(p, \mathring{f}, i)$. By (6), for each $m \geq i+1$, $s^q_m \subset \bigcup_{n \in \left[k^i_{(t \cup v)}, \infty\right)} s^p_n$. Note that $m^q_{(t \cup v)} \leq i < i+1$ and $k^i_{(t \cup v)} > m^p_{(t \cup v)}$ by definition. Since $q \leq_0 p$, it follows that $q(t \cup v, i+1) \leq_0 p\left(t \cup v, k^i_{(t \cup v)}\right)$. Since there exists $x \in \mathbf{V}_{\omega_1}$ such that $p\left(t \cup v, k^i_{(t \cup v)}\right) \Vdash_1 \mathring{f}(i) = x$, this verifies (3).

Finally, we show how to get such $m_0, \ldots, m_i \in \omega$ for each $i \in \omega$. Fix $i \in \omega$, and assume that s^q_j for $j < i$ are given to us. First, fix $k_0 \in \omega$ such that for each $0 \leq n \leq i$ and for each $k \geq k_0$, $\mathrm{nor}^{q_n}_k\left(s^{q_n}_k\right) \geq i$ and $\forall j < i \left[\mathrm{nor}(s^q_j) < \mathrm{nor}^{q_n}_k\left(s^{q_n}_k\right)\right]$. Also fix $l_0 \geq i$ such that for all $j < i$, $s^q_j \subset \bigcup_{m \in [0,l_0)} s^p_m$. Recall that p satisfies $(\dagger_4)$ with respect to $\mathring{f}$. Applying $(\dagger_4)$ to l_0, find $k_1 > l_0$ as in $(\dagger_4)$. Next, choose $k_2 \geq k_1$ such that for each $j < i$, $t \subset \bigcup_{m \in [0,j)} s^q_m$, and $v \subset s^q_j$ such that $t \cup v \in B(p, \mathring{f}, j)$, $k_2 \geq k^j_{(t \cup v)}$. Finally, recall that for each $0 \leq n \leq i$, $q_n \leq_0 p$. So it is possible to choose $k_3 \geq k_0$ such that for each $0 \leq n \leq i$ and each $k \geq k_3$, $s^{q_n}_k \subset \bigcup_{n \in [k_2, \infty)} s^p_n$. Now choose $m_0, \ldots, m_i \geq k_3$. It is easy to see that (4), (6), and (7) are satisfied. For (5), fix $t \subset \bigcup_{m \in [0,i)} s^q_m$, and $u \subset s^q_i$ with $\mathrm{nor}(u) > 0$. Note that $t \subset \bigcup_{m \in [0,k_1)} s^p_m$ and that $i \leq l_0 < k_1$. Moreover, $s^q_i \subset \bigcup_{n \in [k_2, \infty)} s^p_n$. So there exists $m \geq k_2$ such that $\mathrm{nor}^p_m(u \cap s^p_m) > 0$. We have that $m \geq k_2 \geq k_1$, $u \cap s^p_m \subset s^p_m$ and $\mathrm{nor}^p_m(u \cap s^p_m) > 0$. Therefore, by $(\dagger_4)$, there is $v \subset u \cap s^p_m \subset u$ such that $t \cup v \in B(p, \mathring{f}, i)$, and we are done. $\dashv$

Definition 21. A poset $\mathbb{P}$ is said to be *almost* ω^ω*-bounding* if for any $p \in \mathbb{P}$ and $\mathring{f} \in \mathbf{V}^{\mathbb{P}}$ such that $\Vdash \mathring{f} \in \omega^\omega$, there exist $q \leq p$ and $g \in \omega^\omega$ such that for any $X \in [\omega]^\omega$, there exists $q_X \leq q$ such that $q_X \Vdash \exists^\infty n \in X \left[\mathring{f}(n) \leq g(n)\right]$.

It is not difficult to see that an almost ω^ω-bounding poset preserves all σ-directed unbounded families of monotonic functions in ω^ω. Shelah proved that a countable support iteration of proper almost ω^ω-bounding posets does not add a dominating real. He also proved that $\mathbb{P}_0$ is almost ω^ω-bounding (consult either [Sh2] or [Ab]).

Lemma 22. *Assume that $\Vdash_0 \mathscr{A}$ is MAD. Then $\mathbb{P}_1$ is almost ω^ω-bounding.*

Proof. Fix $\mathring{f} \in \mathbf{V}^{\mathbb{P}_1}_{\omega_1}$ such that $\Vdash_1 \mathring{f} \in \omega^\omega$ and $p \in \mathbb{P}_1$. Find $q \leq_0 p$ as in Lemma 20. Define $g \in \omega^\omega$ as follows. For any $k \in \omega$ define $g(k) = \max(X_k)$, where

$$X_k = \left\{ l \in \omega : \exists t \subset \bigcup\nolimits_{m \in [0,k)} s^q_m \exists v \subset s^q_k \left[q(t \cup v, k+1) \Vdash_1 \mathring{f}(k) = l \right] \right\}.$$

Note that X_k is non-empty and finite, so $g(k)$ is well-defined. Now, fix $X \in [\omega]^\omega$ and let q_X be defined as in the proof of Lemma 20. Then $q_X \in \mathbb{P}_1$ and $q_X \leq q$. Fix $r \leq q_X$ and $n \in \omega$. Fix $k^* \geq n$ such that $t = s^r \setminus s^q \subset \bigcup_{m \in [0,k^*)} s^q_m$. Choose $i \in \omega$ such that $s^r_i \subset \bigcup_{m \in [k^*, \infty)} s^q_m$. There must be $k \in X$ with $k \geq k^*$ such that $\mathrm{nor}^q_k(s^q_k \cap s^r_i) > 0$. It follows that there exists $l \in X_k$ and $v \subset s^q_k \cap s^r_i$ such that $q(t \cup v, k+1) \Vdash_1 \mathring{f}(k) = l$. But it is clear that $r(v, i+1) \leq q(t \cup v, k+1)$. So $r(v, i+1) \leq r$ and $r(v, i+1) \Vdash_1 \mathring{f}(k) = l \leq g(k)$. Since $k \in X$ and $k \geq n$, we are done. $\dashv$

We now have all the lemmas needed to give a proof of

Theorem 23. *It is consistent to have $\aleph_1 = \mathfrak{b} < \mathfrak{a}_{closed} = \aleph_2$.*

Proof. Start with a ground model satisfying CH. Fixing a book-keeping device to ensure that all names for witnesses to $\mathfrak{a}_{closed} = \aleph_1$ are eventually taken care of, do a CS iteration $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \omega_2 \rangle$ of proper almost ω^ω-bounding posets as follows. At a stage $\alpha < \omega_2$ suppose that $\mathbb{P}_\alpha$ is given. Let G_α be $(\mathbf{V}, \mathbb{P}_\alpha)$-generic. In $\mathbf{V}[G_\alpha]$, let $\langle X^\alpha_\xi : \xi < \omega_1 \rangle$ be a sequence of non-empty closed subsets of $[\omega]^\omega$ given by the book-keeping device. If $\mathscr{A} = \bigcup_{\xi < \omega_1} X^\alpha_\xi$ is not MAD, then let $\mathbb{Q}_\alpha$ be the trivial poset. Now assume that $\mathscr{A}$ is MAD. Let $\mathbb{C}_{\omega_1}$ be the poset for adding ω_1 Cohen reals. Let H be $(\mathbf{V}[G_\alpha], \mathbb{C}_{\omega_1})$-generic. If $\mathscr{A}$ is not MAD in $\mathbf{V}[G_\alpha][H]$, then in $\mathbf{V}[G_\alpha]$, let $\mathbb{Q}_\alpha = \mathbb{C}_{\omega_1}$. Suppose $\mathscr{A}$ is MAD in $\mathbf{V}[G_\alpha][H]$. In $\mathbf{V}[G_\alpha][H]$, if there exists $p \in \mathbb{P}_0$ such that $p \Vdash_0 \mathscr{A}$ is not MAD, then let $\mathbb{R} = \{q \in \mathbb{P}_0 : q \leq p\}$. If $\Vdash_0 \mathscr{A}$ is MAD, then let $\mathbb{R} = \mathbb{P}_1$ (defined with respect to $\mathscr{A}$). In either case, in $\mathbf{V}[G_\alpha]$ let $\mathring{\mathbb{R}}$ be a full $\mathbb{C}_{\omega_1}$ name for $\mathbb{R}$. Let $\mathbb{Q}_\alpha = \mathbb{C}_{\omega_1} * \mathring{\mathbb{R}}$. Note that in all of these cases $\Vdash_{\mathbb{Q}_\alpha} \mathscr{A}$ is not MAD. In $\mathbf{V}$, let $\mathring{\mathbb{Q}}_\alpha$ be a full $\mathbb{P}_\alpha$ name for $\mathbb{Q}_\alpha$. This completes the definition of the iteration. If G_{ω_2} is $(\mathbf{V}, \mathbb{P}_{\omega_2})$ generic, then since $\mathbb{P}_{\omega_2}$ does not add a dominating real, $\mathfrak{b} = \omega_1$ in $\mathbf{V}[G_{\omega_2}]$. Suppose for a contradiction that $\langle X_\xi : \xi < \omega_1 \rangle$ is a sequence of non-empty closed subsets of $[\omega]^\omega$ such that $\mathscr{A} = \bigcup_{\xi < \omega_1} X_\xi$ is MAD. For some $\alpha < \omega_2$, the book-keeping device ensured that $\langle X_\xi : \xi < \omega_1 \rangle$ was considered at stage α. So there is a set $c \in [\omega]^\omega \cap \mathbf{V}[G_{\alpha+1}]$ such that in $\mathbf{V}[G_{\alpha+1}]$, for each $\xi < \omega_1$, c is almost disjoint from every element of X_ξ. For any fixed $\xi < \omega_1$, this statement is Π^1_1 and hence absolute. So in $\mathbf{V}[G_{\omega_2}]$, for any $\xi < \omega_1$, c is almost disjoint from every element of X_ξ. This is a contradiction. $\dashv$

3. A characterization of $\mathbb{P}_0$

In this section we show that the poset $\mathbb{P}_0$ defined in Section 2, which was used by Shelah in [Sh1] to produce the first consistency proof of $\mathfrak{b} < \mathfrak{s}$, can be viewed as a two step iteration of a countably closed forcing followed by a σ-centered poset.

Definition 24. Let $\mathbb{F} = \{\mathcal{F} : \mathcal{F}$ is a proper F_σ filter on $\omega\}$. Recall our convention that all filters are required to contain the Fréchet filter. We order $\mathbb{F}$ by $\supset$. It is clear that $\mathbb{F}$ is countably closed and adds an ultrafilter on ω. Let $\mathring{\mathcal{U}}$ denote the canonical $\mathbb{F}$-name for the ultrafilter added by $\mathbb{F}$. For any filter $\mathcal{U}$, let $\mathbb{M}(\mathcal{U})$ denote the Mathias-Prikry forcing with $\mathcal{U}$.

In this section we will prove that $\mathbb{P}_0$ is forcing equivalent to $\mathbb{F} * \mathbb{M}(\mathring{\mathcal{U}})$. This is entirely analogous to the characterization of Mathias forcing as first adding a selective ultrafilter with $\mathcal{P}(\omega)/\mathrm{FIN}$ and then doing Mathias-Prikry forcing with that selective ultrafilter. Note that $\mathcal{P}(\omega)/\mathrm{FIN}$ is forcing equivalent to the partial order of all countably generated filters on ω ordered by $\supset$. So $\mathbb{F}$ is a natural generalization of $\mathcal{P}(\omega)/\mathrm{FIN}$. Our first lemma is rather well-known.

Lemma 25. *Let $\mathcal{F}$ be a proper F_σ filter on ω. There is a non-empty closed set $C \subset \mathcal{P}(\omega)$ such that $C \subset \mathcal{F}$ and $\forall b \in \mathcal{F} \exists c \in C\, [c \subset^* b]$.*

Proof. Write $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{T}_n$, where each $\mathcal{T}_n$ is a closed subset of $\mathcal{P}(\omega)$. Let $C = \{b \cup n : n \in \omega \wedge b \in \mathcal{T}_n\}$. It is clear that $\forall b \in \mathcal{F} \exists c \in C\, [c \subset^* b]$ and that $C \subset \mathcal{F}$. Note also that $\omega \in C$. We will check that C is closed. Suppose $\langle c_i : i \in \omega\rangle$ is a sequence of elements of C converging to some $c \in \mathcal{P}(\omega)$. For each $i \in \omega$ fix $n_i \in \omega$ and $b_i \in \mathcal{T}_{n_i}$ such that $c_i = b_i \cup n_i$. By passing to a subsequence, we may assume that the b_i converge to some $b \in \mathcal{P}(\omega)$ and that either $\forall i \in \omega\, [n_i < n_{i+1}]$ or there is a fixed $n \in \omega$ such that $\forall i \in \omega\, [n_i = n]$. In the first case $c = \omega$, and so $c \in C$. In the second case, each $b_i \in \mathcal{T}_n$ and so $b \in \mathcal{T}_n$. $c = b \cup n$, whence $c \in C$. ⊣

Theorem 26. *There is a dense embedding of $\mathbb{P}_0$ into $\mathbb{F} * \mathbb{M}(\mathring{\mathcal{U}})$.*

Proof. Most of the tools needed to prove this have already been developed in the proof of Lemma 9. Fix $p \in \mathbb{P}_0$. Let A_p, C_p, and $\mathcal{F}_p$ be as in Definition 4. As observed in Section 2, $\mathrm{int}(p) \in \mathcal{F}_p$. It follows that $\mathcal{F}_p \Vdash_{\mathbb{F}} \mathrm{int}(p) \in \mathring{\mathcal{U}}$, and so $\langle \mathcal{F}_p, \langle s^p, \mathrm{int}(p)\rangle\rangle$ is a condition in $\mathbb{F} * \mathbb{M}(\mathring{\mathcal{U}})$. Define a map $\phi : \mathbb{P}_0 \to \mathbb{F} * \mathbb{M}(\mathring{\mathcal{U}})$ by $\phi(p) = \langle \mathcal{F}_p, \langle s^p, \mathrm{int}(p)\rangle\rangle$. We will check that ϕ is a dense embedding.

First suppose that $q \le p$. We must show that $\phi(q) \le \phi(p)$. Note that $s^q \supset s^p$, $\mathrm{int}(q) \subset \mathrm{int}(p)$, and that $s^q \setminus s^p \subset \mathrm{int}(p)$. So it suffices to show that $\mathcal{F}_q \supset \mathcal{F}_p$. For this, suppose that $s \in A_q$. Then there is $n \in \omega$ such that $\mathrm{nor}^q_n(s \cap s^q_n) > 0$. As $q \le p$, there must be $m \in \omega$ such that $\mathrm{nor}^p_m(s \cap s^q_n \cap s^p_m) > 0$. Therefore, $\mathrm{nor}^p_m(s \cap s^p_m) > 0$, and so $s \in A_p$. So $A_q \subset A_p$, whence $\mathcal{F}_q \supset \mathcal{F}_p$.

Next, fix $p, q \in \mathbb{P}_0$ and suppose that $\phi(p)$ and $\phi(q)$ are compatible in $\mathbb{F}*\mathbb{M}(\mathring{\mathcal{U}})$. We must show that p and q are compatible. Indeed, we will prove something stronger. Let $\langle \mathcal{F}, \langle s^*, d\rangle\rangle$ be an arbitrary tuple where:

1. $\mathcal{F}$ is an F_σ filter containing both $\mathcal{F}_p$ and $\mathcal{F}_q$;
2. $s^* \in [\omega]^{<\omega}$, $s^* \supset s^p$, $s^* \supset s^q$, $s^* \setminus s^p \subset \mathrm{int}(p)$, and $s^* \setminus s^q \subset \mathrm{int}(q)$;
3. $d \in \mathcal{F}$ and $\forall i \in s^* \forall j \in d\, [i < j]$;
4. $d \subset \mathrm{int}(p) \cap \mathrm{int}(q)$.

We will show that there is $r \in \mathbb{P}_0$ such that $r \le p$, $r \le q$, and $\phi(r) \le \langle \mathcal{F}, \langle s^*, d\rangle\rangle$. The argument that $\phi''\mathbb{P}_0$ is dense in $\mathbb{F} * \mathbb{M}(\mathring{\mathcal{U}})$ is almost identical; so this is enough to finish the proof. Using Lemma 25, find a non-empty closed set $C \subset \mathcal{P}(\omega)$ such that $C \subset \mathcal{F}$ and $\forall b \in \mathcal{F} \exists c \in C\, [c \subset^* b]$. Put

$$A = \{s \in [\omega]^{<\omega} : s \in A_p \cap A_q \wedge \forall c \in C\, [|s \cap c| > 1]\}.$$

We note a few properties of A. It is clear that for each $s \in A$, $|s| > 1$ and that if $t \supset s$, then $t \in A$. Next, fix $b \in \mathcal{F}^+$. For any $c \in C$, $b \cap c \in \mathcal{F}^+$. Therefore, there exist $s \in A_p$ and $\bar{s} \in A_q$ such that $s \subset b \cap c$ and $\bar{s} \subset b \cap c$. By a compactness argument, this implies that there is a finite set $s \subset b$ such that for each $c \in C$, there exists $t \subset s$ such that $t \in A_p$ and $t \subset b \cap c$, and also there exists $\bar{t} \subset s$ such that $\bar{t} \in A_q$ and $\bar{t} \subset b \cap c$. Recall that for any $t \in A_p$, $|t| > 1$. Therefore, for any $c \in C$, $|s \cap c| > 1$. Moreover, since C is non-empty, there are $t \subset s$ and $\bar{t} \subset s$ with $t \in A_p$ and $\bar{t} \in A_q$. Therefore, $s \in A_p \cap A_q$. Thus we have shown that for $b \in \mathcal{F}^+$, there exists $s \subset b$ such that $s \in A$. Lastly, note that for any $c \in C$, there is no $s \in A$ such that $s \subset (\omega \setminus c)$.

Now, let $\mathrm{nor} : [\omega]^{<\omega} \to \omega$ be the norm induced by A, defined exactly as in the proof of Lemma 9. It is easy to check that nor is well-defined and that it is a norm on ω. Just as in the proof of Lemma 9, it is not hard to show by induction on n that for any $b \in \mathcal{F}^+$ there exists $s \subset b$ such that $\mathrm{nor}(s) \geq n$. Define r as follows. $s^r = s^*$. Let n_p be the least $n \in \omega$ such that for all $m \geq n$, $s^p_m \cap s^* = 0$, and let n_q be analogously defined for q. Clearly, $d \cap \left(\bigcup_{m \in [n_p, \infty)} s^p_m\right) \cap \left(\bigcup_{m \in [n_q, \infty)} s^q_m\right) \in \mathcal{F}$. So find $s^r_0 \subset d \cap \left(\bigcup_{m \in [n_p, \infty)} s^p_m\right) \cap \left(\bigcup_{m \in [n_q, \infty)} s^q_m\right)$ with $\mathrm{nor}(s^r_0) > 0$. Now, suppose that s^r_n is given to us with $s^r_n \subset d \cap \left(\bigcup_{m \in [n_p, \infty)} s^p_m\right) \cap \left(\bigcup_{m \in [n_q, \infty)} s^q_m\right)$ and $\mathrm{nor}(s^r_n) > 0$. Put $n^*_p = \max\{m \in \omega : s^r_n \cap s^p_m \neq 0\}$ and $n^*_q = \max\{m \in \omega : s^r_n \cap s^q_m \neq 0\}$. Note that $n_p \leq n^*_p$ and that $n_q \leq n^*_q$. Again, it is clear that $d \cap \left(\bigcup_{m \in [n^*_p+1, \infty)} s^p_m\right) \cap \left(\bigcup_{m \in [n^*_q+1, \infty)} s^q_m\right) \in \mathcal{F}$. So it is possible to find s^r_{n+1} with $\mathrm{nor}(s^r_{n+1}) > \mathrm{nor}(s^r_n)$ such that $s^r_{n+1} \subset d \cap \left(\bigcup_{m \in [n^*_p+1, \infty)} s^p_m\right) \cap \left(\bigcup_{m \in [n^*_q+1, \infty)} s^q_m\right)$. This completes the construction of the s^r_n. For each $n \in \omega$, put $c^r_n = \langle s^r_n, \mathrm{nor} \restriction s^r_n \rangle$ and define $r = \langle s^r, \langle c^r_n : n \in \omega \rangle \rangle$. Observe that for any $s \in [\omega]^{<\omega}$, if $\mathrm{nor}(s) > 0$, then $s \in A$, and so $s \in A_p \cap A_q$, and hence there exist $m, n \in \omega$ such that $\mathrm{nor}^p_m(s \cap s^p_m) > 0$ and $\mathrm{nor}^q_n(s \cap s^q_n) > 0$. It follows that $r \leq p$ and $r \leq q$. It remains to be seen that $\phi(r) = \langle \mathcal{F}_r, \langle s^r, \mathrm{int}(r) \rangle \rangle \leq \langle \mathcal{F}, \langle s^*, d \rangle \rangle$. First suppose that $s \in A_r$. Then by definition, for some $n \in \omega$, $\mathrm{nor}^r_n(s \cap s^r_n) > 0$. Hence $s \in A$, and so $A_r \subset A$. So for any $c \in C$, $\neg \exists s \in A_r$ such that $s \subset \omega \setminus c$. So $c \in C_r$. Thus $C \subset C_r$. It follows that $\mathcal{F} \subset \mathcal{F}_r$. Since $s^r = s^*$ and $\mathrm{int}(r) \subset d$, it follows that $\phi(r) \leq \langle \mathcal{F}, \langle s^*, d \rangle \rangle$. $\dashv$

We make some remarks on how to get an analogous characterization for $\mathbb{P}_1$. Let $\mathscr{A}$ be as in Section 2. Let $\mathbf{V}_{\omega_1}$ be the extension gotten by adding ω_1 Cohen reals. Then in $\mathbf{V}_{\omega_1}$ it is possible to prove that $\mathbb{P}_1$ (defined relative to $\mathscr{A}$) densely embeds into $\mathbb{F}_{\mathscr{A}} * \mathbb{M}(\mathring{\mathcal{U}})$, where $\mathbb{F}_{\mathscr{A}} = \{\mathcal{F} : \mathcal{F}$ is a proper F_σ filter on ω and $\mathcal{I}(\mathscr{A}) \cap \mathcal{F} = 0\}$, ordered by $\supset$, and where $\mathring{\mathcal{U}}$ is the canonical $\mathbb{F}_{\mathscr{A}}$ name for the ultrafilter added by it. The proof of this is nearly identical to the proof of Theorem 26, except that in the construction of r, the Cohen reals must be used like in the proof of Lemma 9. Now, it is easy to see that both in the case of $\mathbb{P}_0$ and in the case of $\mathbb{P}_1$, for the corresponding $\mathbb{M}(\mathring{\mathcal{U}})$ to have the right properties, it is not necessary for $\mathring{\mathcal{U}}$ to be fully generic for $\mathbb{F}$ or $\mathbb{F}_{\mathscr{A}}$ respectively. It is enough to have ultrafilters that are sufficiently generic for $\mathbb{F}$ and $\mathbb{F}_{\mathscr{A}}$. We elaborate on this idea in the next section to give a ccc proof of the consistency of $\mathfrak{b} < \mathfrak{a}_{closed}$.

4. A ccc proof

In this section, we provide a ccc proof of the consistency of $\mathfrak{b} < \mathfrak{a}_{\text{closed}}$. Unlike the proof in Section 2, this proof generalizes to the situation where $\mathfrak{c}$ is larger than ω_2.

Let κ be a regular uncountable cardinal, assume $\mathfrak{c} = \kappa$, $\langle f_\alpha : \alpha < \kappa \rangle$ is a well-ordered unbounded family in ω^ω, and $\langle X_\alpha : \alpha < \lambda \rangle$ is a sequence of non-empty closed subsets of $[\omega]^\omega$ such that $\mathscr{A} = \bigcup_{\alpha<\lambda} X_\alpha$ is a MAD family. Here $\omega_1 \leq \lambda \leq \kappa$. Let $\mathbf{V}_\kappa$ be the extension of $\mathbf{V}$ by adding κ Cohen reals. Assume that (the reinterpretation of) $\mathscr{A}$ is still MAD in $\mathbf{V}_\kappa$.

Theorem 27. *There is an ultrafilter $\mathcal{U}$ extending $\mathcal{F}(\mathscr{A})$ such that $\mathbb{M}(\mathcal{U})$ preserves the unboundedness of $\langle f_\alpha : \alpha < \kappa \rangle$ and forces that (the reinterpretation of) $\mathscr{A}$ is not MAD anymore.*

Proof. The proof of the theorem follows closely the proof of the analogous result for $\mathfrak{a}$ instead of $\mathfrak{a}_{\text{closed}}$, [Br, Theorem 3.1]. However, some of the combinatorics developed for $\mathfrak{a}_{\text{closed}}$ in Section 2 will be needed as well.

Say $\mathcal{F}$ is an $F_{<\kappa}$ filter if it is the union of $< \kappa$ many closed subsets of $[\omega]^\omega$. It is easy to see that the appropriate generalizations of Lemmas 5 and 7 hold.

Lemma 28. *In $\mathbf{V}_\kappa$, let $\mathcal{F}$ be any $F_{<\kappa}$ filter and suppose that $\mathcal{G}$, the filter generated by $\mathcal{F} \cup \mathcal{F}(\mathscr{A})$, is a proper filter. Then $\mathcal{G}$ is P^+.*

Lemma 29. *In $\mathbf{V}_\kappa$, suppose that $\mathcal{F}$ is a $F_{<\kappa}$ filter such that $\mathcal{G}$, the filter generated by $\mathcal{F} \cup \mathcal{F}(\mathscr{A})$, is proper. Suppose $b \in \mathcal{G}^+$. Then for each $\alpha_0, ..., \alpha_k < \omega_1$, there is a $c \in [b]^\omega$ such that $c \in \mathcal{G}^+$ and $\forall (a_0, ..., a_k) \in X_{\alpha_0} \times ... \times X_{\alpha_k} [|(a_0 \cup ... \cup a_k) \cap c| < \omega]$.*

We distinguish two cases. They correspond to the cases where we force with the partial orders $\mathbb{P}_0$ and $\mathbb{P}_1$, respectively, in Section 2, and also to the two cases of the proof of [Br, Theorem 3.1].

Case 1. In $\mathbf{V}_\kappa$, there is a $F_{<\kappa}$ filter $\mathcal{F}$ such that $\mathcal{F}(\mathscr{A}) \subseteq \mathcal{F}$. This corresponds to the situation where we force with $\mathbb{P}_0$ in Section 2. Since this case is different from the corresponding case in [Br], we provide details. Recall [Br, p. 192] that a partial map $\tau : [\omega]^{<\omega} \times \omega \to \omega$ is a *preterm.* If $\mathcal{G} \supseteq \mathcal{F}$ is a filter and $\mathring{g}$ is an $\mathbb{M}(\mathcal{G})$-name for a function in ω^ω, then $\tau = \tau_{\mathring{g}}$ given by $\tau(s, n) = k$ iff (s, G) forces "$\mathring{g}(n) = k$" for some $G \in \mathcal{G}$ is a preterm, the *preterm associated with* $\mathring{g}$. Let $\{\tau_\alpha : \alpha < \kappa\}$ enumerate the set of all preterms. Let $\mathcal{F}_0 = \mathcal{F}$. Recursively build an increasing chain of $F_{<\kappa}$ filters $\mathcal{F}_\alpha$, $\alpha < \kappa$, such that:

- for all $\alpha < \lambda$ there is $b \in \mathcal{F}_{\alpha+1}$ such that $|b \cap a| < \omega$ for all $a \in X_\alpha$;
- if τ_α looks like a name for $\mathcal{F}_\alpha$, then there is $\beta < \kappa$ such that for all filters $\mathcal{H}$ extending $\mathcal{F}_{\alpha+1}$, $\mathbb{M}(\mathcal{H})$ forces that $f_\beta \not\leq^* \mathring{g}$ where $\tau_{\mathring{g}} = \tau_\alpha$;
- if τ_α does not look like a name for $\mathcal{F}_\alpha$, then τ_α is not a preterm associated with any $\mathbb{M}(\mathcal{H})$-name $\mathring{g}$, for any filter $\mathcal{H}$ extending $\mathcal{F}_{\alpha+1}$.

Here we say that τ_α *looks like a name for* $\mathcal{F}_\alpha$ if for all n, all $s \in [\omega]^{<\omega}$, and all $c \in \mathcal{F}_\alpha^+$, there are $t \in [\omega]^{<\omega}$ and $u \subseteq s \cup t$ such that $(u, n) \in \text{dom}(\tau_\alpha)$ and $t \subseteq c$.

For limit ordinals α we simply let $\mathcal{F}_\alpha = \bigcup_{\beta<\alpha} \mathcal{F}_\beta$. So assume $\alpha + 1$ is a successor ordinal. Suppose τ_α does not look like a name for $\mathcal{F}_\alpha$. Then there are n, $s \in [\omega]^{<\omega}$, and $c \in \mathcal{F}_\alpha^+$ witnessing this. That is, whenever $t \subseteq c$ is finite, then for no $u \subseteq s \cup t$ does (u, n) belong to $\text{dom}(\tau_\alpha)$. Let $\mathcal{F}'_\alpha$ be the filter generated by $\mathcal{F}_\alpha$ and c. Then τ_α is not a preterm associated with any $\mathbb{M}(\mathcal{H})$-name $\mathring{g}$, for any filter $\mathcal{H}$ extending $\mathcal{F}'_\alpha$, because no condition compatible with (s, c) would decide $\mathring{g}(n)$.

So suppose τ_α looks like a name for $\mathcal{F}_\alpha$. Assume $\mathcal{F}_\alpha = \bigcup_{\gamma<\mu} K_\gamma$ where $\mu < \kappa$ and all K_γ are compact. Fix γ and fix $T = \{s_j : j < \ell\} \subseteq [\omega]^{<\omega}$. Now define $f = f_{\gamma,T}$ by

$$\begin{aligned} f(n) = \quad & \min\{k : \text{ given } c \in K_\gamma \text{ and } b_j \text{ with } c \subseteq \textstyle\bigcup_{j<\ell} b_j \\ & \text{there are } j < \ell, t \subset b_j, \text{ and } u \subseteq s_j \cup t \text{ with } \tau_\alpha(u,n) \leq k\}. \end{aligned}$$

Let us first check that f is well-defined. Fix $c \in K_\gamma$ and b_j with $c \subseteq \bigcup_{j<\ell} b_j$. Let j be minimal such that $b_j \in \mathcal{F}_\alpha^+$. Since τ_α looks like a name for $\mathcal{F}_\alpha$, there are finite $t \subseteq b_j$ and $u \subseteq s_j \cup t$ such that $(u,n) \in \mathrm{dom}(\tau_\alpha)$. Choose such t and u so that the value $k(\{c, b_j : j < \ell\}) := \tau_\alpha(u,n)$ is minimal. Since $(2^\omega)^\ell$ and K_γ are compact, it is easy to see that the function sending $\langle c, b_j : j < \ell\rangle$ to $k(\{c, b_j : j < \ell\})$ is bounded. Hence f is well-defined.

Now choose β such that $f_\beta \not\leq^* f_{\gamma,T}$ for all $\gamma < \mu$ and finite $T \subseteq [\omega]^{<\omega}$. For $s \in [\omega]^{<\omega}$ and $b \in [\omega]^\omega$ define $g = g_{s,b}$ by

$$g(n) = \min\{k : \exists \text{ finite } t \subseteq b\ \exists u \subseteq s \cup t\ (\tau_\alpha(u,n) = k)\},$$

in case the set on the right-hand side is non-empty; otherwise put $g(n) = \omega$. Let $\mathcal{F}'_\alpha$ be the filter generated by $\mathcal{F}_\alpha$ and all sets of the form $\{\omega \setminus b : \exists s\ (g_{s,b} \geq^* f_\beta)\}$. It is clear that these sets are a union of countably many compact sets.

We first verify that $\mathcal{F}'_\alpha$ still is a proper filter. Suppose this were not the case. Then, for $c \in \mathcal{F}_\alpha$ and sets b_j, $j < \ell$, we would have $\omega \setminus b_j \in \mathcal{F}'_\alpha$ and $c \cap \bigcap_{j<\ell} \omega \setminus b_j = \emptyset$, i.e., $c \subseteq \bigcup_{j<\ell} b_j$. Fix γ such that $c \in K_\gamma$ and s_j such that $g_{s_j,b_j} \geq^* f_\beta$. Set $T = \{s_j : j < \ell\}$. Fix m such that $g_{s_j,b_j}(n) \geq f_\beta(n)$ for all $n \geq m$. By construction there is $n \geq m$ such that $f_{\gamma,T}(n) < f_\beta(n)$. By definition of $f_{\gamma,T}$, there are $j < \ell$, $t \subseteq b_j$, and $u \subseteq s_j \cup t$ with $\tau_\alpha(u,n) \leq f_{\gamma,T}(n)$. But then $g_{s_j,b_j}(n) \leq f_{\gamma,T}(n) < f_\beta(n) \leq g_{s_j,b_j}(n)$, a contradiction.

Next we check that $\mathcal{F}'_\alpha$ is as required. Let $\mathcal{H}$ be any filter extending $\mathcal{F}'_\alpha$, and let $(s,b) \in \mathbb{M}(\mathcal{H})$. Suppose $\mathring{g}$ is $\mathbb{M}(\mathcal{H})$-name such that $\tau_\alpha = \tau_{\mathring{g}}$. Assume there is m such that (s,b) forces $\mathring{g}(n) \geq f_\beta(n)$ for all $n \geq m$. Then clearly $g_{s,b}(n) \geq f_\beta(n)$ for all $n \geq m$. So $\omega \setminus b \in \mathcal{F}'_\alpha \subseteq \mathcal{H}$, a contradiction.

Finally, by Lemma 29, we may find $b \in (\mathcal{F}'_\alpha)^+$ such that $|b \cap a| < \omega$ for all $a \in X_\alpha$. Let $\mathcal{F}_{\alpha+1}$ be the filter generated by $\mathcal{F}'_\alpha$ and b. This completes the recursive construction and Case 1 of the proof.

Case 2. In $\mathbf{V}_\kappa$, there is no $F_{<\kappa}$ filter $\mathcal{F}$ such that $\mathcal{F}(\mathscr{A}) \subseteq \mathcal{F}$. This corresponds to the situation when we force with $\mathbb{P}_1$ in Section 1. This is the more difficult case. However, unlike for Case 1, the proof of [Br] can be taken over almost verbatim in this case. Simply mix applications of Lemma 29 with the recursive construction expounded in [Br, pp. 192-195].

This completes the proof of the theorem. ⊣

Using finite support iteration we now obtain

Theorem 30. *Let κ be a regular uncountable cardinal. It is consistent that $\mathfrak{b} \leq \kappa$ and $\mathfrak{a}_{\text{closed}} = \mathfrak{c} = \kappa^+$.*

5. Tail splitting, club splitting and closed almost disjointness

Definition 31. Let κ be a regular cardinal, and let $\bar{A} = \langle a_\alpha : \alpha < \kappa\rangle \subseteq [\omega]^\omega$. $\bar{A}$ is *tail-splitting* if for every $b \in [\omega]^\omega$ there is $\alpha < \kappa$ such that a_β splits b for all $\beta \geq \alpha$. $\bar{A}$ is *club-splitting* if for every $b \in [\omega]^\omega$, $C_b = \{\alpha < \kappa : a_\alpha \text{ splits } b\}$ contains a club.

Clearly, a tail-splitting sequence is club-splitting, and the existence of a club-splitting sequence of length κ implies that $\mathfrak{s}_\omega \leq \kappa$. Moreover, it is easy to see that $\kappa \leq \mathfrak{r}$, where $\mathfrak{r}$ is the reaping number. In the next section we shall come back to the question of which of these implications reverse.

Definition 32. $\bar{A} = \langle a_{\alpha,n} : \alpha < \kappa, n < \omega \rangle$ is a *tail-splitting sequence of partitions* if the $a_{\alpha,n}$, $n \in \omega$, are pairwise disjoint and for all $b \in [\omega]^\omega$ there is α such that $a_{\beta,n}$ splits b for all $\beta \geq \alpha$ and all $n \in \omega$. Similarly, $\bar{A}$ is a *club-splitting sequence of partitions* if for all $b \in [\omega]^\omega$, $C_b = \{\alpha < \kappa : \text{all } a_{\alpha,n} \text{ split } b\}$ contains a club.

Clearly a tail-splitting sequence of partitions yields a tail-splitting sequence, but we don't know whether the converse is true (see Question 48). Similarly for club-splitting.

We begin with two observations:

Observation 33. *In the Hechler model (the model obtained by adding at least ω_2 Hechler reals over a model of CH), there is a tail-splitting sequence of partitions of length ω_1.*

To see this, notice that the classical proof, of the consistency of $\mathfrak{s} < \mathfrak{b}$, due to Baumgartner and Dordal [BD], shows that tail-splitting sequences of partitions from the ground model are preserved in the iterated Hechler extension.

Observation 34. $\mathfrak{d} = \aleph_1$ *implies the existence of a tail-splitting sequence of partitions of length ω_1.*

Definition 35. Say there is a *splitting sequence of partitions over models* if there are $\bar{M} = \langle M_\alpha : \alpha < \omega_1 \rangle$ and $\bar{A} = \langle a_{\alpha,n} : \alpha < \omega_1, n < \omega \rangle$ such that:

- $\bar{M}$ is a strictly increasing continuous sequence of countable models of a large enough fragment of ZFC;
- for each α, $\langle a_{\alpha,n} : n \in \omega \rangle$ is pairwise disjoint, belongs to $M_{\alpha+1}$, and all $a_{\alpha,n}$ split all members of M_α;
- whenever $b \in [\omega]^\omega$, there are α and a model N of a large enough fragment of ZFC containing b such that $M_\alpha \subseteq N$, $N \cap M = M_\alpha$, and all $a_{\alpha,n}$ split all members of N.

Here, $M = \bigcup_{\alpha<\omega_1} M_\alpha$.

Lemma 36. *The existence of a club-splitting sequence of partitions of length ω_1 implies the existence of a splitting sequence of partitions over models.*

Proof. Assume $\bar{B} = \langle b_{\alpha,n} : \alpha < \omega_1, n < \omega \rangle$ is a club-splitting sequence of partitions. Let χ be a large enough regular cardinal. Let $\bar{M} = \langle M_\alpha : \alpha < \omega_1 \rangle$ be such that for each $\alpha < \omega_1$:

(1) $\bar{B} \in M_0$, $M_\alpha \prec H(\chi)$, $|M_\alpha| = \omega$, and $M_\alpha \in M_{\alpha+1}$;
(2) if α is a limit, then $M_\alpha = \bigcup_{\xi<\alpha} M_\xi$.

For each $\alpha < \omega_1$, let $\delta_\alpha = M_\alpha \cap \omega_1$. Define $\langle a_{\alpha,n} : n \in \omega \rangle = \langle b_{\delta_\alpha,n} : n \in \omega \rangle$. For any $\alpha < \omega_1$ and $x \in [\omega]^\omega \cap M_\alpha$, there is a club $C \in M_\alpha$ such that for all $\delta \in C$ and $n \in \omega$, $b_{\delta,n}$ splits x. As $\delta_\alpha \in C$, $a_{\alpha,n}$ splits x for all $n \in \omega$. Next, if $b \in [\omega]^\omega$, then let $N \prec H(\chi)$ be countable with $\bar{M} \in N$ and $b \in N$. Let $\gamma = N \cap \omega_1$. It is clear that $N \cap \left(\bigcup_{\xi<\omega_1} M_\xi \right) = M_\gamma$ and moreover, $\gamma = \delta_\gamma$. Again, for any $x \in [\omega]^\omega \cap N$ there is a club $C \in N$ such that for all $\delta \in C$ and $n \in \omega$, $b_{\delta,n}$ splits x. As $\gamma = \delta_\gamma \in C$, we are done. ⊣

Theorem 37. *The existence of a splitting sequence of partitions over models implies* $\mathfrak{a}_{\text{closed}} = \aleph_1$.

Proof. This follows from a straightforward analysis of the proof of [BK, Lemma 3.4]. Since the proof of the latter lemma is rather long and technical, we will not repeat it here and simply stress the main points. We assume the reader to have a copy of [BK] at hand.

Assume we are at stage α, and closed sets $A_\beta \in M_\alpha$ have been constructed so that $\bigcup_{\beta<\alpha} A_\beta$ is an almost disjoint family. (We do not assume that the whole sequence of the A_β belongs to M_α; this does not matter.) The A_β are obtained as sets of branches through a tree whose levels form a partition of a subset of ω. Now, from the $a_{\alpha,n}$, one obtains a sequence C_σ^Θ of pairwise disjoint subsets of ω, where $\sigma \in \omega^{<\omega}$ and Θ comes from a certain set of finite sequences of finite sequences, which is used to construct the next set A_α. To obtain the C_σ^Θ from the $a_{\alpha,n}$, one has to remove finitely many elements (the "excluded points") as well as a set from M_α (the set X_σ), see the end of part 1 in the proof of [BK, Lemma 3.4] for details. Obviously, the resulting C_σ^Θ will still split all $Y \in M_\alpha$ such that $Y \setminus X_\sigma$ is infinite, and this is all that's needed for the rest of the proof to go through. This completes the construction of the A_α. We need to check they are as required.

Part 2 of the proof of [BK, Lemma 3.4] does not apply, and steps 1 and 2 of part 3 carry over without any change. The heart of the proof is step 3 of part 3 (the last part of the proof), namely, the argument showing that $\bigcup_{\beta<\omega_1} A_\beta$ is indeed maximal. Take any $Y \in \omega^\omega$. Find α and N such that they satisfy the last clause of Definition 35 for $b = Y$. Now, as in the proof of [BK, Lemma 3.4], build functions $g_j \in \omega^\omega \cap N$ and a decreasing sequence of subsets $Y_j \in N$ of Y. This is possible because $M_\alpha \subseteq N$. (Again, we do not require that the sequences of the g_j or Y_j belong to N, but this is not needed.) Assume that Y is almost disjoint from all elements of A_β, for $\beta < \alpha$. Using the g_j and Y_j a function h is constructed such that the branch in A_α associated with h is a subset of Y, i.e. there is $a \in A_\alpha$ with $a \subseteq Y$. For the construction of h, the splitting properties of the C_σ^Θ together with the fact that any initial segment of h is constructed in N are used. $\dashv$

Using the theorem, we obtain two results from the literature as corollaries.

Corollary 38 (Brendle and Khomskii, [BK])**.** *In the Hechler model,* $\mathfrak{a}_{\text{closed}} = \aleph_1$. *In particular,* $\mathfrak{b} > \mathfrak{a}_{\text{closed}}$ *is consistent.*

Corollary 39 (Raghavan and Shelah, [RS])**.** $\mathfrak{d} = \aleph_1$ *implies* $\mathfrak{a}_{\text{closed}} = \aleph_1$.

6. Tail splitting: a consistency result

In this section, we show that the existence of a tail-splitting sequence is not the same as the existence of a club-splitting sequence in the sense of Definition 31.

Theorem 40. *It is consistent that there is a club-splitting family of size* $\aleph_1$ *and there is no tail-splitting family of size* $\aleph_1$. *In particular,* $\mathfrak{s} = \aleph_1$.

Assume $\bar{A} = \langle a_\alpha : \alpha < \omega_1 \rangle$ is club-splitting. Let $\mathbb{P}$ be a forcing notion. Say that $\mathbb{P}$ preserves $\bar{A}$ if $\bar{A}$ is still club-splitting in the $\mathbb{P}$-generic extension. It is easy to see that if $(\mathbb{P}_\alpha : \alpha < \delta)$ is an fsi of ccc forcing and all $\mathbb{P}_\alpha$ $(\alpha < \delta)$ preserve $\bar{A}$, then so does $\mathbb{P}_\delta$.

Also let $\mathcal{H}$ be a filter on ω. We say that $(\star)_{\bar{A},\mathcal{H}}$ holds if for every partial function $f : \omega \to \omega$ with $\mathrm{dom}(f) \in \mathcal{H}^+$ and $f^{-1}(\{n\}) \in \mathcal{H}^*$, the set $D_f = \{\alpha < \omega_1 : f^{-1}(a_\alpha)$ and $f^{-1}(\omega \setminus a_\alpha)$ both belong to $\mathcal{H}^+\}$ contains a club.

Lemma 41. *Assume $(\star)_{\bar{A},\mathcal{H}}$ holds. Then $\mathbb{L}(\mathcal{H})$ preserves $\bar{A}$.*

Proof. Let $\mathring{a}$ be an $\mathbb{L}(\mathcal{H})$-name for an infinite subset of ω. We need to find a club set $C \subset \omega_1$ in the ground model such that the trivial condition forces that a_α splits $\mathring{a}$ for all $\alpha \in C$. We can assume that $\mathring{a}$ is thin in the sense that the increasing enumeration $\mathring{g}$ of $\mathring{a}$ is forced to dominate the generic Laver real $\mathring{\ell}$.

We briefly recall the standard rank analysis of Laver forcing $\mathbb{L}(\mathcal{H})$. Let φ be a formula. For any $s \in \omega^{<\omega}$, say that s *forces* φ if there is a condition with stem s which forces φ. Say that s *favors* φ if s does not force $\neg\varphi$. Define the rank function rk_φ by induction:

- $\mathrm{rk}_\varphi(s) = 0$ iff s forces φ;
- $\mathrm{rk}_\varphi(s) \le \alpha$ iff there is $c \in \mathcal{H}^+$ such that $\mathrm{rk}_\varphi(s\hat{\ }n) < \alpha$ for all $n \in c$;
- $\mathrm{rk}_\varphi(s) = \alpha$ iff $\mathrm{rk}_\varphi(s) \le \alpha$ but $\mathrm{rk}_\varphi(s) \not\le \beta$ for $\beta < \alpha$.

A standard argument shows that s favors φ iff $\mathrm{rk}_\varphi(s) < \omega_1$. (Suppose $\mathrm{rk}_\varphi(s)$ is undefined. Then one constructs a tree $T \in \mathbb{L}(\mathcal{H})$ with stem s such that for all nodes $t \in T$ extending s, $\mathrm{rk}_\varphi(t)$ is undefined. In particular, no extension of s in T has rank 0, and therefore T must force $\neg\varphi$. Thus s does not favor φ. Suppose, on the other hand, that s forces $\neg\varphi$. We prove by induction on α that $\mathrm{rk}_\varphi(s) > \alpha$. This is obvious for $\alpha = 0$. So assume $\alpha > 0$. Let $T \in \mathbb{L}(\mathcal{H})$ be a tree with stem s witnessing that s forces $\neg\varphi$. Let $c \in \mathcal{H}$ be the successor level of s in T. By induction hypothesis $\mathrm{rk}_\varphi(s\hat{\ }n) \ge \alpha$ for all $n \in c$. By definition of the rank, we see that $\mathrm{rk}_\varphi(s) > \alpha$.)

Say that $s \in \omega^{<\omega}$ is *good for n* if s does not favor $\mathring{g}(n) = k$ for any k, but $\{m : s\hat{\ }m$ favors $\mathring{g}(n) = k$ for some $k\}$ is $\mathcal{H}$-positive.

Claim 41.1. If $|s| \le n$ and $\mathrm{stem}(T) = s$, then there is $t \in T$ extending s which is good for n.

Proof. Define a new rank function ρ by stipulating:

- $\rho(t) = 0$ if t favors $\mathring{g}(n) = k$ for some k;
- $\rho(t) \le \alpha$ iff there is $c \in \mathcal{H}^+$ such that $\rho(t\hat{\ }n) < \alpha$ for all $n \in c$.

Notice that $\rho(s) < \omega_1$. (Otherwise there would be a tree $T' \in \mathbb{L}(\mathcal{H})$ with stem s such that all nodes of T' extending s have undefined rank. Now find $t \in T'$ extending s and forcing $\mathring{g}(n) = k$ for some k. Clearly $\rho(t) = 0$, a contradiction.) On the other hand, $|s| \le n$ and $\mathring{g} \ge \mathring{\ell}$ imply that $\rho(s) \ge 1$ because for each k there is a tree T' with stem s forcing $\mathring{\ell}(n) > k$ and, hence, $\mathring{g}(n) > k$. Thus we can find $t \in T$ extending s such that $\rho(t) = 1$. By definition, this means that t does not favor $\mathring{g}(n) = k$ for any k, and that $\{m : t\hat{\ }m$ favors $\mathring{g}(n) = k$ for some $k\}$ belongs to $\mathcal{H}^+$. ⊣

For each node s which is good for n, define a partial function $f_{s,n}$ by letting $\mathrm{dom}(f_{s,n}) = \{m : s\hat{\ }m$ favors $\mathring{g}(n) = k$ for some $k\}$ and setting $f_{s,n}(m) = k$ for some k such that $s\hat{\ }m$ favors $\mathring{g}(n) = k$, for $m \in \mathrm{dom}(f_{s,n})$. Note that such k is not necessarily unique, but this does not matter. By definition of goodness, it is immediate that $f_{s,n}$ satisfies the stipulations in the definition of $(\star)_{\bar{A},\mathcal{H}}$, i.e.

$\mathrm{dom}(f_{s,n}) \in \mathcal{H}^+$ and $f_{s,n}^{-1}(\{k\}) \in \mathcal{H}^*$ for all k. Now let C be the intersection of all $D_{f_{s,n}}$ where s is good for n. We show that C is as required.

Claim 41.2. The trivial condition forces that a_α splits $\mathring{a}$ for all $\alpha \in C$.

Proof. Let T be any condition and n_0 a natural number. We need to find $n, n' \geq n_0$ and $T' \leq T$ such that T' forces $\mathring{g}(n) \in a_\alpha$ and $\mathring{g}(n') \notin a_\alpha$. Since the proofs are identical, we only produce n. Let s be the stem of T. Choose $n \geq n_0, |s|$. By the previous claim, there is $t \in T$ extending s which is good for n. Hence $f_{t,n}$ is defined. Since $\alpha \in D_{f_{t,n}}$, $f_{t,n}^{-1}(a_\alpha)$ belongs to $\mathcal{H}^+$. Hence we can find $m \in \mathrm{dom}(f_{t,n})$ in the successor level of t in T such that $k := f_{t,n}(m) \in a_\alpha$. Since $t\hat{\ }m$ favors $\mathring{g}(n) = k$, there is a subtree T' of T with stem extending $t\hat{\ }m$ which forces $\mathring{g}(n) = k$. Therefore T' forces $\mathring{g}(n) \in a_\alpha$, as required. ⊣

This completes the proof of the lemma. ⊣

Lemma 42. *Assume CH. Assume $\bar{B} = \langle b_\alpha : \alpha < \omega_1 \rangle$ is tail-splitting. Then there is $\{c_\alpha : \alpha < \omega_1\}$ such that $c_\alpha \subseteq b_{\zeta_\alpha}$ for some $\zeta_\alpha \geq \alpha$ and the c_α generate a P-filter $\mathcal{H}$ such that $(\star)_{\bar{A},\mathcal{H}}$ holds.*

Proof. Let $\{f_\alpha : \alpha < \omega_1\}$ list all partial finite-to-one functions $\omega \to \omega$. Recursively we find $\subseteq^*$-decreasing $c_\alpha \in [\omega]^\omega$, $\zeta_\alpha \geq \alpha$, continuous increasing γ_α, and decreasing club sets C_α such that:

- $c_\alpha \subseteq^* b_{\zeta_\alpha}$,
- $\gamma_\alpha \in C_\alpha$,

and for all $\beta < \alpha$ such that $\mathrm{dom}(f_\beta) \cap c_\beta$ is infinite the following hold:

- a_{γ_δ} splits $f_\beta(c_\alpha)$ (i. e. $f_\beta^{-1}(a_{\gamma_\delta}) \cap c_\alpha$ and $f_\beta^{-1}(\omega \setminus a_{\gamma_\delta}) \cap c_\alpha$ are both infinite) for $\beta < \delta \leq \alpha$;
- for all $\gamma \in C_\alpha$, a_γ splits the sets $f_\beta(c_\alpha)$.

Basic step: $c_0 = b_0$, $\zeta_0 = 0$, $\gamma_0 = 0$.

Successor step: $\alpha \to \alpha + 1$. Since $\bar{B}$ is tail-splitting, we can find $\zeta_{\alpha+1} \geq \alpha + 1$ such that $b_{\zeta_{\alpha+1}}$ splits all sets $f_\beta^{-1}(a_{\gamma_\delta}) \cap c_\alpha$ and $f_\beta^{-1}(\omega \setminus a_{\gamma_\delta}) \cap c_\alpha$ for $\beta < \delta \leq \alpha$, as well as $\mathrm{dom}(f_\alpha) \cap c_\alpha$ if the latter set is infinite. In particular, the intersection of $b_{\zeta_{\alpha+1}}$ with these sets is infinite. Let $c_{\alpha+1} = c_\alpha \cap b_{\zeta_{\alpha+1}}$. Then a_{γ_δ} splits $f_\beta(c_{\alpha+1})$ for $\beta < \delta < \alpha + 1$. Since $\bar{A}$ is club-splitting, there is a club set $C_{\alpha+1} \subseteq C_\alpha$ such that for all $\gamma \in C_{\alpha+1}$, a_γ splits all sets $f_\beta(c_{\alpha+1})$ for $\beta < \alpha$, as well as $f_\alpha(c_{\alpha+1})$ in case $\mathrm{dom}(f_\alpha) \cap c_\alpha$ is infinite. Now let $\gamma_{\alpha+1}$ be the least element of $C_{\alpha+1}$ greater than γ_α.

Limit step: α limit. Let $C' = \bigcap\{C_\beta : \beta < \alpha\}$. Let $\gamma_\alpha = \bigcup\{\gamma_\beta : \beta < \alpha\}$. Clearly $\gamma_\alpha \in C'$. So a_{γ_α} splits all $f_\beta(c_\delta)$ where $\beta < \delta < \alpha$. Construct c' as a pseudo-intersection of c_δ, $\delta < \alpha$, such that all a_{γ_δ} still split all $f_\beta(c')$ for $\beta < \delta \leq \alpha$.

Since $\bar{B}$ is tail-splitting, we can find $\zeta_\alpha \geq \alpha$ such that b_{ζ_α} splits all sets $f_\beta^{-1}(a_{\gamma_\delta}) \cap c'$ and $f_\beta^{-1}(\omega \setminus a_{\gamma_\delta}) \cap c'$ for $\beta < \delta \leq \alpha$. Let $c_\alpha = c' \cap b_{\zeta_\alpha}$. Since $\bar{A}$ is club-splitting, we can find $C_\alpha \subseteq C'$ club with $\gamma_\alpha \in C_\alpha$ and such that for all $\gamma \in C_\alpha$, a_γ splits the sets $f_\beta(c_\alpha)$.

This completes the recursive construction. We need to show that the c_α are as required. Clearly, they generate a P-filter $\mathcal{H}$. Let $f : \omega \to \omega$ be a partial function with $\mathrm{dom}(f) \in \mathcal{H}^+$ and $f^{-1}(\{n\}) \in \mathcal{H}^*$ for all n. Since $\mathcal{H}$ is a P-filter, the sets $f^{-1}(\omega \setminus n)$ have a pseudo-intersection $A \in \mathcal{H}$. Notice that the restriction of f to A is finite-to-one. So we may assume without loss of generality that f is finite-to-one.

Hence there is β such that $f = f_\beta$. Since $\mathrm{dom}(f_\beta) \in \mathcal{H}^+$, $\mathrm{dom}(f_\beta) \cap c_\beta$ is clearly infinite. By construction, for all $\alpha > \beta$ and all $\delta > \beta$, a_{γ_δ} splits $f_\beta(c_\alpha)$. Hence both $f_\beta^{-1}(a_{\gamma_\delta})$ and $f_\beta^{-1}(\omega \setminus a_{\gamma_\delta})$ are $\mathcal{H}$-positive. Thus the club set $D_f = D_{f_\beta} = \{\gamma_\delta : \delta > \beta\}$ is as required. $\dashv$

We finally discuss an application of tail-splitting.

Definition 43. The *strong polarized partition relation* $\begin{pmatrix} \lambda \\ \kappa \end{pmatrix} \to \begin{pmatrix} \lambda \\ \kappa \end{pmatrix}^{1,1}_2$ means that for every function $c : \lambda \times \kappa \to 2$ there are $A \subseteq \lambda$ and $B \subseteq \kappa$ of size λ and κ, respectively, such that $c \restriction (A \times B)$ is constant.

The following was essentially observed by Garti and Shelah [GS2, Claim 1.3], though they stated this in somewhat different language.

Observation 44. *The following are equivalent:*

(1) $\begin{pmatrix} \lambda \\ \omega \end{pmatrix} \to \begin{pmatrix} \lambda \\ \omega \end{pmatrix}^{1,1}_2$*;*

(2) $cf(\lambda) \neq \omega$ *and there does not exist a tail-splitting sequence of length* λ*.*

In particular, Garti and Shelah [GS1, Claim 1.4] observed that $\mathfrak{s} > \aleph_1$ implies that $\begin{pmatrix} \omega_1 \\ \omega \end{pmatrix} \to \begin{pmatrix} \omega_1 \\ \omega \end{pmatrix}^{1,1}_2$ holds. As a consequence of Theorem 40, we obtain:

Corollary 45. *It is consistent that* $\mathfrak{s} = \aleph_1$ *and* $\begin{pmatrix} \omega_1 \\ \omega \end{pmatrix} \to \begin{pmatrix} \omega_1 \\ \omega \end{pmatrix}^{1,1}_2$ *holds.*

This answers [GS3, Question 1.7(a)].

7. Open problems

We conclude with a number of open problems. Perhaps the most interesting is:

Question 46. *Does* $\mathfrak{s} = \aleph_1$ *(or at least* $\mathfrak{s}_\omega = \aleph_1$*) imply* $\mathfrak{a}_{\mathrm{closed}} = \aleph_1$*?*

While the existence of a tail-splitting sequence of length ω_1 is strictly stronger than the existence of a club-splitting sequence of length ω_1 (Theorem 40), we in fact do not know whether the latter is stronger than $\mathfrak{s}_\omega = \aleph_1$ or $\mathfrak{s} = \aleph_1$.

Question 47. *Is it consistent that* $\mathfrak{s} = \aleph_1$ *(or even* $\mathfrak{s}_\omega = \aleph_1$*) and there is no club-splitting sequence of length* ω_1*?*

For the proof of $\mathfrak{a}_{\mathrm{closed}} = \aleph_1$ we needed a club-splitting sequence of partitions (Lemma 36 and Theorem 37). It is unclear whether a club-splitting sequence is enough. In fact, we do not know whether the two notions are equivalent.

Question 48. *Does the existence of a tail-splitting sequence of length* κ *imply the existence of a tail-splitting sequence of partitions of length* κ*? Similarly for club-splitting instead of tail-splitting.*

Let $\mathfrak{a}_{\mathrm{Borel}}$ denote the size of the smallest family $\mathscr{A}$ a.d. Borel sets such that $\bigcup \mathscr{A}$ is mad. Clearly, $\aleph_1 \leq \mathfrak{a}_{\mathrm{Borel}} \leq \mathfrak{a}_{\mathrm{closed}}$. We do not know, however, whether the cardinals are equal.

Question 49 (Brendle and Khomskii [BK, Question 4.7])**.** *Is* $\mathfrak{a}_{\mathrm{Borel}} = \mathfrak{a}_{\mathrm{closed}}$*?*

If this is not the case one could ask

Question 50 (Brendle and Khomskii [BK, Question 4.4]). *Is* $\mathfrak{b} < \mathfrak{a}_{\mathrm{Borel}}$ *consistent?*

Finally we address

Question 51 (see also [BK, Conjecture 4.5]). *Is* $\mathfrak{h} \leq \mathfrak{a}_{\mathrm{closed}}$*? Or even* $\mathfrak{h} \leq \mathfrak{a}_{\mathrm{Borel}}$*?*

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