SUPERSATURATED IDEALS

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Dedicated to the memory of Ken Kunen

ABSTRACT. An ideal \mathcal{I} on a set X is supersaturated iff $\operatorname{add}(\mathcal{I}) \geq \omega_2$ and for every family \mathcal{F} of \mathcal{I} -positive sets with $|\mathcal{F}| < \operatorname{add}(\mathcal{I})$, there exists a countable set that meets every set in \mathcal{F} . We show that many well-known ccc forcings preserve supersaturation. We also show that the existence of supersaturated ideals is independent of ZFC plus "There exists an ω_1 -saturated σ -ideal".

1. INTRODUCTION

Saturation properties of ideals are ubiquitous in modern set theory and there is a considerable body of work (for example, see [3, 5, 6, 7]) on the study of a large number of such properties. Throughout this paper, by an ideal \mathcal{I} on X, we mean an ideal \mathcal{I} on X that contains every finite subset of X. Supersaturation is a strengthening of ω_1 -saturation defined as follows.

Definition 1.1. Suppose \mathcal{I} is an ideal on X and λ is a cardinal. We say that \mathcal{I} is λ -supersaturated iff $\mathsf{add}(\mathcal{I}) \geq \lambda^+$ and for every $\mathcal{A} \subseteq \mathcal{I}^+$, if $|\mathcal{A}| < \mathsf{add}(\mathcal{I})$, then there exists $W \in [X]^{<\lambda}$ such that for every $A \in \mathcal{A}$, $A \cap W \neq \emptyset$. \mathcal{I} is supersaturated iff it is ω_1 -supersaturated.

Suppose \mathcal{I} is a supersaturated ideal on X. Since $\operatorname{add}(\mathcal{I}) \geq \omega_2$, it follows that \mathcal{I}^+ cannot have an uncountable subfamily of pairwise disjoint sets because no countable set can meet all of them. So \mathcal{I} is ω_1 -saturated. Let $\mu = \operatorname{add}(\mathcal{I})$. Ulam showed that either μ is a measurable cardinal or μ is a weakly inaccessible cardinal $\leq \mathfrak{c}$. Solovay showed that μ admits a normal ω_1 -saturated ideal \mathcal{J} and μ is a measurable cardinal in the inner model $L[\mathcal{J}]$. For proofs of these facts, see [7].

Though closely related to some of the works of Fremlin, supersaturated ideals were formally introduced in [4] where it was shown that if $\kappa \leq \mathfrak{c}$ admits a normal supersaturated ideal then the order dimension of the Turing degrees is at least κ . An earlier motivation for investigating these ideals comes from the following question of Fremlin – See Problem EG(h) in [1].

Question 1.2 (Fremlin). Suppose κ is real valued measurable and $m : \mathcal{P}(\kappa) \to [0,1]$ is a witnessing normal measure. Let \mathcal{F} be a family of subsets of κ such that $|\mathcal{F}| < \kappa$ and for every $A \in \mathcal{F}$, m(A) > 0. Must there exist a countable $N \subseteq \kappa$ such that for every $A \in \mathcal{F}$, $N \cap A \neq \emptyset$?

So Question 1.2 is asking if the null ideal of every normal witnessing measure on a real valued measurable cardinal must be supersaturated. One of the standard ways of obtaining ω_1 -saturated ideals on cardinals below the continuum is to start

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with a measurable cardinal κ and a witnessing normal prime ideal \mathcal{I} on κ , and force with a ccc forcing \mathbb{P} that adds $\geq \kappa$ reals. Let \mathcal{J} be the ideal generated by \mathcal{I} in $V^{\mathbb{P}}$. Then \mathcal{J} is always an ω_1 -saturated normal ideal on $\kappa \leq \mathfrak{c}$. But whether or not \mathcal{J} is supersaturated will depend on the choice of \mathbb{P} . This motivates the notion of supersaturation preserving forcings (Definition 2.1). In Section 2, we show that a large class of ccc forcings for adding new reals are supersaturation preserving. In particular, the following holds.

Theorem 1.3. Let $Random_{\lambda}$ denote the forcing for adding λ random reals.

- (1) Every σ -linked forcing is supersaturation preserving.
- (2) Random_{λ} is supersaturation preserving for every λ .

The question of whether every ω_1 -saturated ideal must be supersaturated was raised in [4]. Our main result shows that this is independent.

Theorem 1.4. Each of the following is consistent.

- (1) There is an ω_1 -saturated ideal on a cardinal below the continuum and there are no supersaturated ideals.
- (2) There is an ω_1 -saturated ideal on a cardinal below the continuum and every ω_1 -saturated ideal is supersaturated.

Notation: Let \mathcal{I} be an ideal on X. Define $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$. $\mathsf{add}(\mathcal{I})$ denotes the least cardinality of a subfamily of \mathcal{I} whose union is in \mathcal{I}^+ . For $A \subseteq X$, define $\mathcal{I} \upharpoonright A = \{Y \subseteq X : Y \cap A \in \mathcal{I}\}$. Suppose $V \subseteq W$ are transitive models of set theory, $X, \mathcal{I} \in V$ and $V \models ``\mathcal{I}$ is an ideal on X''. Recall that the ideal generated by \mathcal{I} in W is $\mathcal{J} = \{A \in W : (\exists B \in \mathcal{I})(A \subseteq B)\}$.

For a set of ordinals X, $\operatorname{otp}(X)$ denotes the order type of X. An ordinal δ is indecomposable iff for every $X \subseteq \delta$, either $\operatorname{otp}(X) = \delta$ or $\operatorname{otp}(\delta \setminus X) = \delta$. If \mathbb{P} , \mathbb{Q} are forcing notions, we write $\mathbb{P} < \mathbb{Q}$ iff $\mathbb{P} \subseteq \mathbb{Q}$ and every maximal antichain in \mathbb{P} is also a maximal antichain in \mathbb{Q} . Cohen_{λ} denotes the forcing for adding λ Cohen reals. Random_{λ} is the measure algebra on 2^{λ} equipped with the usual product measure denoted by μ_{λ} . If λ is clear from the context, then we drop it and just write μ .

2. CCC Forcings and supersaturation

Definition 2.1. A forcing \mathbb{P} is κ -ssp (ssp = supersaturation preserving) iff for every normal supersaturated ideal \mathcal{I} on κ , $V^{\mathbb{P}} \models$ "the ideal generated by \mathcal{I} is supersaturated". \mathbb{P} is ssp iff it is κ -ssp for every κ .

In [4], the following forcings were shown to be κ -ssp for every κ .

- (a) Cohen_{λ} for any λ .
- (b) Any finite support iteration of ccc forcings of size $< \kappa$.

It was also shown that $\mathsf{Random}_{\lambda}$ is κ -ssp for any measurable κ . The next theorem improves this to all κ .

Theorem 2.2. Random_{λ} is κ -ssp for every κ and λ .

Proof. Fix a normal supersaturated ideal \mathcal{I} on κ . Put $\mathbb{B} = \mathsf{Random}_{\lambda}$ and let \mathcal{J} be the ideal generated by \mathcal{I} in $V^{\mathbb{B}}$. Suppose $\theta < \kappa$ and $\Vdash_{\mathbb{B}} \langle \mathring{A}_i : i < \theta \rangle$ is a sequence of \mathcal{J} -positive sets. It suffices to find $B \in [\kappa]^{\aleph_0}$ such that $\Vdash_{\mathbb{B}} (\forall i < \theta) (\mathring{A}_i \cap B \neq \emptyset)$.

For $i < \theta$ and $\alpha < \kappa$, put $p_{i,\alpha} = [[\alpha \in \mathring{A}_i]]_{\mathbb{B}}$. So each $p_{i,\alpha}$ is a Baire subset of 2^{λ} . Put $T_i = \{\alpha < \kappa : p_{i,\alpha} \neq 0_{\mathbb{B}}\}.$ Claim 2.3. For each $p \in \mathbb{B} \setminus \{0_{\mathbb{B}}\}, \{\alpha \in T_i : p_{i,\alpha} \cap p \neq 0_{\mathbb{B}}\} \in \mathcal{I}^+$.

Proof. Put $X_p = \{ \alpha \in T_i : p_{i,\alpha} \cap p \neq 0_{\mathbb{B}} \}$ and suppose $X_p \in \mathcal{I}$. Since the empty condition forces that $\mathring{A}_i \in \mathcal{J}^+$, it follows that for every $X \in \mathcal{I}$, $\{p_{i,\alpha} : \alpha \in T_i \setminus X\}$ is predense in \mathbb{B} . But every condition in $\{p_{i,\alpha} : \alpha \in T_i \setminus X_p\}$ is incompatible with p which is impossible. \square

For a finite partial function f from λ to 2, define $[f] = \{x \in 2^{\lambda} : x \mid \mathsf{dom}(f) = f\}.$ For a clopen $K \subseteq 2^{\lambda}$, define supp(K) to be the smallest finite set $S \subseteq \lambda$ such that $(\forall x, y \in 2^{\lambda})(x \upharpoonright \overline{S} = y \upharpoonright S \implies (x \in K \iff y \in K)).$ If supp(K) = S, then there there is finite list $\{f_{K,n} : n < n_{\star}\}$ where $f_{K,n}$'s are pairwise distinct functions from S to 2 and $K = \bigsqcup_{n < n_*} [f_{K,n}].$

Definition 2.4. Suppose C is a family of clopen sets in 2^{λ} . We say that C is a strong Δ -system of width (n_{\star}, N_{\star}) iff $n_{\star}, N_{\star} < \omega$ and the following hold.

- (a) $\langle supp(K) : K \in \mathcal{C} \rangle$ is a Δ -system with root R.
- (b) For every $K \in \mathcal{C}$, $|supp(K) \setminus R| = n_{\star}$.
- (c) For every $K \in \mathcal{C}$, $K = \bigsqcup_{n < N_{\star}} [f_{K,n}]$ where each $f_{K,n} : supp(K) \to 2$ and $f_{K,n}$'s are pairwise distinct.
- (d) For every $K_1, K_2 \in \mathcal{C}$ and $n < N_{\star}$,

(i) $f_{K_{1,n}} \upharpoonright R = f_{K_{2,n}} \upharpoonright R$ and (ii) if for $m \in \{1, 2\}, \{\xi_j^m : j < |R| + n_\star\}$ lists $supp(K_m)$ in increasing order, then $f_{K_{1,j}}(\xi_j^1) = f_{K_{2,j}}(\xi_j^2)$ for every $j < |R| + n_\star$.

Lemma 2.5. Suppose $p \subseteq 2^{\lambda}$ is Baire and C is an infinite strong Δ -system of clopen sets in 2^{λ} of width (n_{\star}, N_{\star}) . Let $\varepsilon > 0$ and assume that for infinitely many $K \in \mathcal{C}, \ \mu(p \cap K) \geq \varepsilon$. Then for all but finitely many $K \in \mathcal{C}, \ \mu(p \cap K) \geq \varepsilon/2$.

Proof. Let R be the root of $(\operatorname{supp}(K) : K \in \mathcal{C})$. For each $K \in \mathcal{C}$, fix $(f_{K,n} : n < N_{\star})$ such that $K = \bigsqcup_{n < N_{\star}} [f_{K,n}]$. First suppose that p is clopen. Let $\mathcal{C}_p = \{K \in \mathcal{C} :$ $(\operatorname{supp}(K) \setminus R) \cap \operatorname{supp}(p) = \emptyset$. Then $\mathcal{C} \setminus \mathcal{C}_p$ is finite and for each $K \in \mathcal{C}_p$,

$$\mu(p \cap K) = \sum_{n < N_{\star}} \mu(p \cap [f_{K,n}]) = 2^{-n_{\star}} \sum_{n < N_{\star}} \mu(p \cap [f_{K,n} \upharpoonright R])$$

which does not depend on $K \in \mathcal{C}_p$. It follows that the result holds if p is clopen. The general case follows by applying the previous case to a clopen $q \subseteq 2^{\lambda}$ satisfying $\mu(p\Delta q) < \varepsilon/2.$

For each $\alpha \in T_i$, fix $S_{i,\alpha} \in [\lambda]^{\aleph_0}$ such that $p_{i,\alpha}$ is supported in $S_{i,\alpha}$. For every $i < \theta, \alpha \in T_i \text{ and } \varepsilon > 0 \text{ rational, choose a clopen set } K_{i,\alpha,\varepsilon} \subseteq 2^{\lambda} \text{ with } \mathsf{supp}(K_{i,\alpha,\varepsilon}) \subseteq \mathcal{O}$ $S_{i,\alpha}$ such that

$$\frac{\mu(p_{i,\alpha}\Delta K_{i,\alpha,\varepsilon})}{\mu(K_{i,\alpha,\varepsilon})} < \varepsilon$$

Claim 2.6. For each $i < \theta$ and $\varepsilon > 0$ rational, we can find $\mathcal{F}_{i,\varepsilon} \subseteq \mathcal{I}^+$ and $\langle (n_{i,\varepsilon,Y}, N_{i,\varepsilon,Y}) : Y \in \mathcal{F}_{i,\varepsilon} \rangle$ such that the following hold.

- (1) $\mathcal{F}_{i,\varepsilon}$ is a countable family of pairwise disjoint sets and $T_i \setminus \bigcup \mathcal{F}_{i,\varepsilon} \in \mathcal{I}$.
- (2) For each $Y \in \mathcal{F}_{i,\varepsilon}$, $\{K_{i,\alpha,\varepsilon} : \alpha \in Y\}$ is a strong Δ -system of width $(n_{i,\varepsilon,Y}, N_{i,\varepsilon,Y}).$

Proof. Fix $i < \theta$ and $\varepsilon > 0$ rational. To simplify notation, we write K_{α} instead of $K_{i,\alpha,\varepsilon}$. It suffices to show that for every \mathcal{I} -positive $X \subseteq T_i$, there exists $Y \subseteq X$ such that $Y \in \mathcal{I}^+$ and there exist (n_Y, N_Y) such that $\{K_{\alpha} : \alpha \in Y\}$ is a strong Δ -system of width (n_Y, N_Y) . Since then we can take $\mathcal{F}_{i,\varepsilon}$ to be a maximal disjoint family of such Y's. That each $\mathcal{F}_{i,\varepsilon}$ is countable follows from the fact that \mathcal{I} is ω_1 -saturated.

Fix a club $E \subseteq \kappa$ such that for every $\gamma \in E$ and $\alpha \in T_i \cap \gamma$, $\max(\operatorname{supp}(K_\alpha)) < \gamma$. Suppose $X \subseteq T_i \cap E$ and $X \in \mathcal{I}^+$. Since \mathcal{I} is normal and the map $\alpha \mapsto \max(\operatorname{supp}(K_\alpha \cap \alpha))$ is regressive on X, we can find $R \subseteq \kappa$ finite and $Y_1 \subseteq X$ such that $Y_1 \in \mathcal{I}^+$, $(\forall \alpha \in Y_1)(\operatorname{supp}(K_\alpha) \cap \alpha = R)$ and $|\operatorname{supp}(K_\alpha) \setminus R| = n_*$ does not depend on $\alpha \in Y_1$. It also follows that $\langle \operatorname{supp}(K_\alpha) : \alpha \in Y_1 \rangle$ forms a Δ -system with root R. For each $\alpha \in Y_1$, let $K_\alpha = \bigsqcup_{n < N_\alpha} [f_{\alpha,n}]$ where each $f_{\alpha,n} : \operatorname{supp}(K_\alpha) \to 2$. Choose $Y_2 \subseteq Y_1$ such that $Y_2 \in \mathcal{I}^+$ and $N_\alpha = N_*$ does not depend on $\alpha \in Y_2$. Finally, choose $Y \subseteq Y_2$ such that $Y \in \mathcal{I}^+$ and $\{K_\alpha : \alpha \in Y\}$ is a strong Δ -system of width (n_*, N_*) .

Since \mathcal{I} is supersaturated, we can choose $B \in [\kappa]^{\aleph_0}$ such that for every $i < \theta$, $\varepsilon > 0$ rational and $Y \in \mathcal{F}_{i,\varepsilon}$, we have $|B \cap Y| = \aleph_0$. It suffices to show that for each $i < \theta$, $\{p_{i,\alpha} : \alpha \in B\}$ is predense in \mathbb{B} .

Suppose not. Fix $i < \theta$ and $p \subseteq 2^{\lambda}$ Baire such that $\mu(p) > 0$ and for every $\alpha \in B$, $\mu(p_{i,\alpha} \cap p) = 0$. Let $X = \{\alpha \in T_i : \mu(p_{i,\alpha} \cap p) > 0\}$. By Claim 2.3, $X \in \mathcal{I}^+$. Using the argument in the proof of Claim 2.6, we can choose $\varepsilon > 0$ rational, $X_* \subseteq X$ and $n_*, N_* < \omega$ such that

(a) $X_{\star} \in \mathcal{I}^+$ and for each $\alpha \in X_{\star}$, $\mu(p_{i,\alpha} \cap p) \geq 4\varepsilon$.

(b) $\{K_{i,\alpha,\varepsilon} : \alpha \in X_{\star}\}$ is a strong Δ -system of width (n_{\star}, N_{\star}) .

Choose $Y \in \mathcal{F}_{i,\varepsilon}$ such that $Y \cap X_{\star} \in \mathcal{I}^+$. Since $|Y \cap X_{\star}| \geq \aleph_0$ and $|Y \cap B| = \aleph_0$, by Lemma 2.5, we can choose $\alpha \in Y \cap B$ such that $\mu(p \cap K_{i,\alpha,\varepsilon}) \geq 2\varepsilon$. But since $\mu(p_{i,\alpha}\Delta K_{i,\alpha,\varepsilon}) \leq \varepsilon \mu(K_{i,\alpha,\varepsilon}) \leq \varepsilon$, it follows that $\mu(p \cap p_{i,\alpha}) \geq \varepsilon > 0$: Contradiction. This completes the proof of Theorem 2.2.

Theorem 2.7. Every σ -linked forcing is κ -ssp for every κ .

Proof. Let \mathcal{I} be a normal supersaturated ideal on κ . Suppose \mathbb{P} is a σ -linked forcing and \mathcal{J} is the ideal generated by \mathcal{I} in $V^{\mathbb{P}}$. Fix $\theta < \kappa$ and WLOG, assume that the trivial condition forces that $\langle \mathring{A}_i : i < \theta \rangle$ is a sequence of \mathcal{J} -positive sets. It suffices to construct $X \in [\kappa]^{\aleph_0}$ such that $\Vdash_{\mathbb{P}} (\forall i < \theta)(X \cap \mathring{A}_i \neq \emptyset)$.

Since \mathbb{P} is σ -linked, we can write $\mathbb{P} = \bigcup \{L_n : n < \omega\}$ where each $L_n \subseteq \mathbb{P}$ has pairwise compatible members. For each $i < \theta$ and $n < \omega$, define

$$B_{i,n} = \{ \alpha < \kappa : (\exists p \in L_n) (p \Vdash \alpha \in \mathring{A}_i) \}$$

Claim 2.8. $W_i = \bigcup \{L_n : n < \omega, B_{i,n} \in \mathcal{I}^+\}$ is dense in \mathbb{P} .

Proof. Suppose not and fix $p \in \mathbb{P}$ such that no extension of p lies in W_i . Put $C = \{\alpha < \kappa : (\exists q \leq p)(q \Vdash \alpha \in \mathring{A}_i)\}$. Since no extension of p lies in W_i , it follows that $C \subseteq \bigcup \{B_{i,n} : n < \omega, B_{i,n} \in \mathcal{I}\}$ and hence $C \in \mathcal{I}$. It now follows that $p \Vdash \mathring{A}_i \in \mathcal{J}$ which is impossible. \Box

Since \mathcal{I} is supersaturated, we can find a countable $X \subseteq \kappa$ such that for every $i < \theta$ and $n < \omega$, if $B_{i,n} \in \mathcal{I}^+$, then $X \cap B_{i,n} \neq \emptyset$. We claim that $\Vdash (\forall i < \theta)(X \cap \mathring{A}_i \neq \emptyset)$. Suppose not and fix $p \in \mathbb{P}$ and $i < \theta$ such that $p \Vdash X \cap \mathring{A}_i = \emptyset$. Using Claim 2.8, choose $n < \omega$ and $p' \leq p$ such that $p' \in L_n$ and $B_{i,n} \in \mathcal{I}^+$. Choose $\alpha \in B_{i,n} \cap X$ and $q \in L_n$ such that $q \Vdash \alpha \in \mathring{A}_i$. Since L_n is linked, we can find a common extension $r \in \mathbb{P}$ of p', q. But $r \Vdash \alpha \in X \cap \mathring{A}_i$: Contradiction. \Box

Corollary 2.9. Each of the following forcings is ssp: Cohen, random, Amoeba, Hechler, Eventually different real forcing.

We do not know if we can improve Theorem 2.7 to the class of σ -finite-cc forcings. For example, one can ask the following.

Question 2.10. Suppose \mathbb{B} is a boolean algebra and $m : \mathbb{B} \to [0,1]$ is a strictly positive finitely additive measure on \mathbb{B} . Must \mathbb{B} be supersaturation preserving?

The next two facts are well known.

Fact 2.11. Suppose \mathbb{P} is a separative σ -linked forcing. Then $|\mathbb{P}| \leq \mathfrak{c}$.

Fact 2.12. Let $\langle (\mathbb{P}_{\xi}, \mathring{\mathbb{Q}}_{\xi}) : \xi < \lambda \rangle$ be a finite support iteration with limit \mathbb{P}_{λ} where for every $\xi < \lambda$, $V^{\mathbb{P}_{\xi}} \models \mathring{\mathbb{Q}}_{\xi}$ is σ -linked. Assume $\lambda < \mathfrak{c}^+$. Then \mathbb{P}_{λ} is also σ -linked.

Theorem 2.13. Let \mathcal{I} be a normal supersaturated ideal on κ and let $\lambda \leq \kappa^+$. Suppose $\langle (\mathbb{P}_{\xi}, \mathring{\mathbb{Q}}_{\xi}) : \xi < \lambda \rangle$ is a finite support iteration with limit \mathbb{P}_{λ} where for every $\xi < \lambda$, $V^{\mathbb{P}_{\xi}} \models \mathring{\mathbb{Q}}_{\xi}$ is σ -linked. Let \mathcal{J} be the ideal generated by \mathcal{I} in $V^{\mathbb{P}_{\lambda}}$. Then \mathcal{J} is supersaturated.

Proof. By induction on λ . First suppose $\kappa \leq \mathfrak{c}$. If $\lambda < \kappa^+$, then by Fact 2.12, \mathbb{P}_{λ} is σ -linked and the claim holds by Theorem 2.7. So assume $\lambda = \kappa^+$ and fix any \mathbb{P}_{λ} -generic filter G_{λ} over V. Let $\langle A_i : i < \theta \rangle$ be a sequence of \mathcal{J} -positive sets in $V[G_{\lambda}]$ where $\theta < \kappa$. Since \mathbb{P}_{λ} is a finite support iteration of ccc forcings, there exists $\eta < \lambda = \kappa^+$ such that $\langle A_i : i < \theta \rangle \in V[G_{\eta}]$ where $G_{\eta} = \mathbb{P}_{\eta} \cap G_{\lambda}$. Note that each A_i is \mathcal{J}_{η} -positive where \mathcal{J}_{η} is the ideal generated by \mathcal{I} in $V[G_{\eta}]$. By inductive hypothesis, there is a countable set that meets A_i for every $i < \theta$. Hence \mathcal{J} is supersaturated.

Next assume $\kappa > \mathfrak{c}$. Then κ is measurable and \mathcal{I} is a normal prime ideal on κ . First suppose $\lambda \leq \kappa$. By Fact 2.11, $|\mathbb{P}_{\xi}| \leq |\xi \cdot \mathfrak{c}| < \kappa$ for every $\xi < \kappa$. Hence by Theorem 4.9 in [4], it follows that \mathcal{J} is supersaturated. Next suppose $\kappa < \lambda \leq \kappa^+$. Note that $V^{\mathbb{P}_{\kappa}} \models \mathfrak{c} \geq \kappa$ since Cohen reals are added at each stage of cofinality ω . So we can work in $V^{\mathbb{P}_{\kappa}}$ and repeat the argument for the case $\kappa \leq \mathfrak{c}$.

It is now natural to ask the following.

Question 2.14 ([4]). Suppose κ is measurable. Is every ccc forcing κ -ssp?

In Section 4, we'll show that the answer is negative. We end this section with the following weaker positive result.

Theorem 2.15. Suppose κ is measurable and \mathcal{I} is a normal prime ideal on κ . Let \mathbb{B} be a ccc complete boolean algebra. Then $V^{\mathbb{B}} \models$ "the ideal generated by \mathcal{I} is ω_2 -supersaturated."

Proof. It suffices to show that the following holds in $V^{\mathbb{B}}$: For every $\mathcal{A} \subseteq \mathcal{J}^+$, if $|\mathcal{A}| < \kappa$, then there exists $X \in [\kappa]^{\aleph_1}$ such that X meets every member of \mathcal{A} .

Suppose $\theta < \kappa$ and $\Vdash_{\mathbb{B}} \{ \mathring{A}_i : i < \theta \} \subseteq \mathcal{J}^+$. Choose $Y \subseteq \kappa$ of \mathcal{I} -measure one such that for every $i < \theta$ and $\alpha \in Y$, $p_{i,\alpha} = [[\alpha \in \mathring{A}_i]] > 0_{\mathbb{B}}$. Using the inaccessibility of κ , the following claim is easy to check.

Claim 2.16. There exists $\langle \mathbb{B}_{\alpha} : \alpha < \kappa \rangle$ such that the following hold.

- (i) $\mathbb{B}_{\alpha} \leq \mathbb{B}$ and $|\mathbb{B}_{\alpha}| < \kappa$.
- (ii) \mathbb{B}_{α} 's are increasing and continuous at α when $cf(\alpha) > \aleph_0$.
- (iii) $\{p_{i,\beta} : \beta < \alpha, i < \theta\} \subseteq \mathbb{B}_{\alpha}.$

Let $\pi_{\alpha} : \mathbb{B} \to \mathbb{B}_{\alpha}$ be a projection map witnessing $\mathbb{B}_{\alpha} < \mathbb{B}$. Choose $f : \kappa \to \kappa$ such that for every $i < \theta$ and $\alpha < \kappa$, we have $\alpha < f(\alpha)$ and $p_{i,\alpha} \in \mathbb{B}_{f(\alpha)}$. Choose $Y_1 \subseteq Y$ of measure one and $\alpha_{\star} < \kappa$ such that for every $i < \theta$, $\pi_{\alpha}(p_{i,\alpha}) = p_{i,\star} \in \mathbb{B}_{\alpha_{\star}}$ does not depend on $\alpha \in Y_1$ and range $(f \upharpoonright \alpha) \subseteq \alpha$ for every $\alpha \in Y_1$. Note that $p_{i,\star} = 1_{\mathbb{B}}$ since $\Vdash_{\mathbb{B}} \mathring{A}_i \in \mathcal{J}^+$. Let $X \subseteq Y \setminus \alpha_{\star}$ be such that $\mathsf{otp}(X) = \omega_1$ and for every $\alpha < \beta$ in $X, f(\alpha) < f(\beta)$.

Claim 2.17. For every $i < \theta$, $\{p_{i,\alpha} : \alpha \in X\}$ is predense in \mathbb{B} .

Proof. Let $\sup(X) = \gamma_{\star}$. Then $\mathsf{cf}(\gamma_{\star}) = \aleph_1$ and hence $\mathbb{B}_{\gamma_{\star}} = \bigcup \{\mathbb{B}_{\gamma} : \gamma \in X\}$. Fix $i < \theta$. Given $p \in \mathbb{B}$, choose $\gamma \in X$ such that $\pi_{\gamma_{\star}}(p) \in \mathbb{B}_{\gamma}$. Now since

$$\mathbb{B} = \mathbb{B}_{\gamma} \star \mathbb{B}_{\gamma_{\star}} / \mathbb{B}_{\gamma} \star \mathbb{B} / \mathbb{B}_{\gamma_{\star}}$$

we can decompose $p = (\pi_{\gamma_{\star}}(p), 1, x)$ and $p_{i,\gamma} = (1, y, 1)$. Hence $p, p_{i,\gamma}$ are compatible.

It follows that \mathcal{J} is ω_2 -supersaturated.

3. Consistently, there are ω_1 -saturated ideals on \mathfrak{c} and all of them are supersaturated

The aim of this section is to show that it is consistent that every ω_1 -saturated σ -ideal is supersaturated.

Theorem 3.1. It is consistent that there is a normal supersaturated ideal on \mathfrak{c} and every ω_1 -saturated σ -ideal is supersaturated.

Lemma 3.2. Suppose that every σ -ideal \mathcal{I} satisfying (i)-(iv) below is supersaturated.

- (i) \mathcal{I} is a uniform ideal on λ ,
- (ii) $\mu \leq \lambda$,
- (iii) for every $X \in \mathcal{I}^+$, $\operatorname{add}(\mathcal{I} \upharpoonright X) = \mu$ and
- (iv) \mathcal{I} is ω_1 -saturated.

Then every ω_1 -saturated σ -ideal is supersaturated.

Proof. Suppose \mathcal{J} is an ω_1 -saturated σ -ideal on X. Note that for every $A \in \mathcal{J}^+$, there exists $B \subseteq A$ such that $(\star)_B$ holds where

 $(\star)_B$ says the following: $B \in \mathcal{J}^+$, $[B]^{<|B|} \subseteq \mathcal{J}$ and for every $C \subseteq B$, if $C \in \mathcal{J}^+$, then $\mathsf{add}(\mathcal{J} \upharpoonright C) = \mathsf{add}(\mathcal{J} \upharpoonright B)$.

Since \mathcal{J} is ω_1 -saturated, we can find a countable partition \mathcal{F} of X such that for each $B \in \mathcal{F}$, $(\star)_B$ holds. Now by assumption, each $\mathcal{J} \upharpoonright B$ is supersaturated. Hence \mathcal{J} is also supersaturated.

Lemma 3.3. Suppose \mathbb{P} is a ccc forcing, $\kappa > \mathfrak{c}$ and $V^{\mathbb{P}} \models \mathcal{J}$ is a κ -complete ω_1 saturated uniform ideal on λ . Let $\mathcal{I} = \{X \subseteq \kappa : 1_{\mathbb{P}} \Vdash X \in \mathcal{J}\}$. Then there is a
countable partition \mathcal{F} of λ such that for every $A \in \mathcal{F}, \mathcal{I} \upharpoonright A = \{Y \subseteq \lambda : Y \cap A \in \mathcal{I}\}$ is a κ -complete prime ideal on λ .

Proof. It is clear that \mathcal{I} is a κ -complete uniform ideal on λ . Suppose $\mathcal{F} \subseteq \mathcal{I}^+$ is an uncountable family of pairwise disjoint sets. For each $A \in \mathcal{F}$, choose $p_A \in \mathbb{P}$ such that $p_A \Vdash A \notin \mathcal{J}$. Since \mathbb{P} is ccc, some $p \in \mathbb{P}$ forces uncountably many p_A 's into the \mathbb{P} -generic filter. But this contradicts the fact that \mathcal{J} is ω_1 -saturated in $V^{\mathbb{P}}$. So \mathcal{I} is ω_1 -saturated. Since \mathcal{I} is κ -complete and $\kappa > \mathfrak{c}$, \mathcal{I} is nowhere atomless. Hence there is a countable partition \mathcal{F} of λ such that for every $A \in \mathcal{F}$, $\mathcal{I} \upharpoonright A = \{Y \subseteq \lambda : Y \cap A \in \mathcal{I}\}$ is a κ -complete prime ideal on λ .

Lemma 3.4. Suppose κ is an inaccessible cardinal and \mathcal{U} is a κ -complete uniform ultrafilter on λ . Let $\mathbb{P} = \mathsf{Cohen}_{\kappa}$. Let \mathcal{J} be the ideal generated by the dual ideal of \mathcal{U} in $V^{\mathbb{P}}$. Then for each $\mathcal{A} \subseteq \mathcal{J}^+$, if $|\mathcal{A}| < \kappa$, then there exists a countable set that meets every member of \mathcal{A} .

Proof. We identify conditions $p \in \mathbb{P}$ as members of the Baire algebra on 2^{κ} which is the σ -algebra generated by clopen subsets of 2^{κ} . Note that for every Baire $p \subseteq 2^{\kappa}$ there is a countable $S \subseteq \kappa$ such that for every $x, y \in 2^{\kappa}$ satisfying $x \upharpoonright S = y \upharpoonright S$, we have $x \in p$ iff $y \in p$. We call such an S, a support of p. The ordering on $\operatorname{Cohen}_{\kappa}$ is defined by $p \leq q$ iff $p \setminus q$ is meager in 2^{κ} . Recall that if $p \subseteq 2^{\kappa}$ is Baire and $S \in [\kappa]^{\aleph_0}$ is a support of p then there is a countable family \mathcal{P} of clopen subsets of 2^{κ} each supported in S such that the symmetric difference of p and $\bigcup \mathcal{P}$ is meager. So p is completely determined by the family \mathcal{P} .

It is clear that \mathcal{J} is a κ -complete uniform ideal on λ . Suppose $\theta < \kappa$ and $\langle \mathring{A}_i : i < \theta \rangle$ is a sequence of \mathcal{J} -positive sets in $V^{\mathbb{P}}$. WLOG, assume that the trivial condition forces this. For $i < \theta$ and $\alpha < \lambda$, let $p_{i,\alpha} = [[\alpha \in \mathring{A}_i]]_{\mathbb{P}}$. Note that for each $i < \theta$, and $Z \in \mathcal{U}$, $\{p_{i,\alpha} : \alpha \in Z\}$ is predense in \mathbb{P} since otherwise some condition will force $\mathring{A}_i \in \mathcal{J}$. Since \mathcal{U} is κ -complete, we can choose $X \in \mathcal{U}$ such that for every $i < \theta$ and $\alpha \in X$, $p_{i,\alpha} > 0_{\mathbb{P}}$. Let $S_{i,\alpha} \in [\kappa]^{\aleph_0}$ be a support of $p_{i,\alpha}$. Since κ is inaccessible, we can choose $Y \subseteq X$ such that $Y \in \mathcal{U}$ and for each $i < \theta$, the following hold.

- (a) For every $\alpha, \beta \in Y$, $(S_{i,\alpha}, 2^{S_{i,\alpha}}, p_{i,\alpha}) \cong (S_{i,\beta}, 2^{S_{i,\beta}}, p_{i,\beta})$. Put $\operatorname{otp}(S_{i,\alpha}) = \gamma_i$. Let $h_{i,\alpha} : \gamma_i \to S_{i,\alpha}$ be the order isomorphism and define $H_{i,\alpha} : 2^{\gamma_i} \to 2^{S_{i,\alpha}}$ by $H_{i,\alpha}(x) = x \circ h_{i,\alpha}^{-1}$. Choose $p_i \subseteq 2^{\gamma_i}$ such that $H_{i,\alpha}[p_i] = p_{i,\alpha}$.
- (b) For each $\gamma < \gamma_i$, either $|\{h_{i,\alpha}(\gamma) : \alpha \in Y\}| = 1$ or for every $Z \in \mathcal{U}$, $|\{h_{i,\alpha}(\gamma) : \alpha \in Z \cap Y\}| \ge \kappa$. Put $\Gamma_i = \{\gamma < \gamma_i : |\{h_{i,\alpha}(\gamma) : \alpha \in Y\}| = 1\}$ and $h_{i,\alpha}[\Gamma_i] = R_i$.

Define

$$B_{i,\alpha} = \{ x \in 2^{R_i} : \{ y \upharpoonright (S_{i,\alpha} \setminus R_i) : y \in p_{i,\alpha} \land y \upharpoonright R_i = x \} \text{ is meager} \}.$$

Then $B_{i,\alpha} = B_i$ does not depend on $\alpha \in Y$ and B_i is measured in 2^{R_i} since otherwise $\{p_{i,\alpha} : \alpha \in Y\}$ will not be predense in \mathbb{P} .

Using (b), choose $B \in [Y]^{\aleph_0}$ such that for every $i < \theta$ and $\alpha \neq \beta$ in $B, S_{i,\alpha} \cap S_{i,\beta} = R_i$. It follows now that for every $i < \theta$, $\{p_{i,\alpha} : \alpha \in B\}$ is predense in \mathbb{P} . Hence $\Vdash (\forall i < \theta)(B \cap \mathring{A}_i \neq \emptyset)$. **Proof of Theorem 3.1:** Let $V \models \text{``} \mathfrak{c} = \omega_1$ and κ is the least measurable cardinal". Let $\mathbb{P} = \mathsf{Cohen}_{\kappa}$. We already know that there is a normal supersaturated ideal on $\kappa = \mathfrak{c}$ in $V^{\mathbb{P}}$. Let us check that, $V^{\mathbb{P}} \models \text{``Every } \omega_1$ -saturated σ -ideal is supersaturated". By Lemma 3.2, it suffices to consider ideals \mathcal{J} that satisfy the following for some $\omega_1 \leq \mu \leq \lambda$.

- (i) \mathcal{J} is a uniform ideal on λ ,
- (ii) for every $X \in \mathcal{J}^+$, $\mathsf{add}(\mathcal{J} \upharpoonright X) = \mu$ and
- (ii) \mathcal{J} is ω_1 -saturated.

Since $V^{\mathbb{P}} \models \mathfrak{c} = \kappa$, we can assume that $\mu \leq \kappa$. Otherwise there is a countable partition \mathcal{E} of λ into \mathcal{J} -positive sets such that for each $X \in \mathcal{E}, \mathcal{J} \upharpoonright X$ is a μ -complete prime ideal and it easily follows that \mathcal{J} is supersaturated.

Towards a contradiction, suppose $\mu < \kappa$. Working in $V^{\mathbb{P}}$, define an ideal \mathcal{K} on μ as follows. Since $\mathsf{add}(\mathcal{J}) = \mu$, we can choose a family $\{A_i : i < \mu\} \subseteq \mathcal{J}$ of pairwise disjoint sets such that $\bigcup_{i < \mu} A_i \in \mathcal{J}^+$. Define

$$\mathcal{K} = \{ \Gamma \subseteq \mu : \bigcup \{ A_i : i \in \Gamma \} \in \mathcal{J} \}$$

It is easy to see that \mathcal{K} is a μ -additive ω_1 -saturated ideal on μ . For simplicity, assume that $\mathbb{1}_{\mathbb{P}} \Vdash \mathring{\mathcal{K}}$ is a μ -additive ω_1 -saturated ideal on μ . Coming back to V, define $\mathcal{K}' = \{X \subseteq \mu : \mathbb{1}_{\mathbb{P}} \Vdash X \in \mathring{\mathcal{K}}\}$. It is clear that $V \models \mathcal{K}'$ is a μ -additive ideal on μ . We claim that $V \models \mathcal{K}'$ is ω_1 -saturated. Suppose not and fix $\langle (A_{\xi}, p_{\xi}) : \xi < \omega_1 \rangle$ such that A_{ξ} 's are pairwise disjoint subsets of μ and for every $\xi < \omega_1, p_{\xi} \Vdash A_{\xi} \notin \mathring{\mathcal{K}}$. Since \mathbb{P} is ccc, we can find some $p_* \in \mathbb{P}$ that forces uncountable many p_{ξ} 's into the generic $G_{\mathbb{P}}$. But this means that $p_* \Vdash \mathring{K}$ is not ω_1 -saturated which is impossible. So $V \models \mathcal{K}'$ is ω_1 -saturated. So μ is weakly inaccessible in V. Since $V \models \mu > \omega_1 = \mathfrak{c}$, it follows that μ must be measurable in V. But κ is the least measurable cardinal in V. Hence $\mu \geq \kappa$: Contradiction.

So we must have $\mu = \kappa$. Let $\mathcal{I} = \{Y \subseteq \lambda : \mathbb{1}_{\mathbb{P}} \Vdash X \in \mathcal{J}\}$. By Lemma 3.3, there is a countable partition \mathcal{F} of λ such that for each $X \in \mathcal{F}, \mathcal{I} \upharpoonright X$ is a κ -complete prime ideal on λ . For each $X \in \mathcal{F}$, let \mathcal{I}_X be the ideal generated by $\mathcal{I} \upharpoonright X$ in $V^{\mathbb{P}}$. By Lemma 3.4, for every $\mathcal{A} \subseteq \mathcal{I}_X^+$, if $|\mathcal{A}| < \kappa$, then there is a countable set that meets every member of \mathcal{A} . Since $\mathcal{I}_A \subseteq \mathcal{J} \upharpoonright A$ and $\mathsf{add}(\mathcal{J} \upharpoonright A) = \kappa$, it follows that $\mathcal{J} \upharpoonright A$ is supersaturated for each $A \in \mathcal{F}$. Since \mathcal{F} is a countable partition of λ , it follows that \mathcal{J} is also supersaturated.

4. Killing supersaturated ideals

Definition 4.1. Suppose $\delta < \omega_1$ is indecomposable and κ is an infinite cardinal. Let $\mathbb{Q}^{\kappa}_{\delta}$ consist of all countable partial maps from κ to 2 such that

- (1) $otp(dom(p)) < \delta$ and
- (2) $\{\xi \in dom(p) : p(\xi) = 1\}$ is finite.

For $p, q \in \mathbb{Q}_{\delta}^{\kappa}$ define $p \leq q$ iff $q \subseteq p$. Let \mathbb{P}_{κ} be the finite support product of $\{Q_{\delta}^{\kappa} : \delta < \omega_{1}, \delta \text{ indecomposable}\}.$

Lemma 4.2. Let \mathbb{P}_{κ} be as in Definition 4.1.

- (1) \mathbb{P}_{κ} is ccc.
- (2) If $\kappa \geq \omega_1$, then \mathbb{P}_{κ} is not σ -finite-cc.

Proof. (1) Towards a contradiction, suppose $A = \{p_i : i < \omega_1\}$ is an uncountable antichain in \mathbb{P}_{κ} . Put $D_i = \mathsf{dom}(p_i)$. By passing to an uncountable subset of A, we can assume that D_i 's form a Δ -system with root D. For each $\delta \in D$ and $i < \omega_1$, put $s_{i,\delta} = \{\gamma : p_i(\delta)(\gamma) = 1\}$ and $X_{i,\delta} = \{\gamma : p_i(\delta)(\gamma) = 0\}$. Note that $\mathsf{otp}(X_{i,\delta}) < \delta$. Choose $B \in [A]^{\omega_1}$ such that for each $\delta \in D$, $\langle s_{i,\delta} : i \in B \rangle$ is a Δ -system with root s_{δ} and for every i < j in B, $s_{j,\delta} \cap X_{i,\delta} = \emptyset$.

Choose $j \in B$ and $\delta \in D$ such that letting $C = \{i \in B \cap j : p_i(\delta) \perp_{\mathbb{Q}_{\delta}} p_j(\delta)\}$, every transversal of $\{s_{i,\delta} \setminus s_{\delta} : i \in C\}$ has order type $\geq \delta$. Now observe that $X_{j,\delta}$ has to meet $s_{i,\delta} \setminus s_{\delta}$ for every $i \in C$. Hence $\mathsf{otp}(X_{j,\delta}) \geq \delta$: Contradiction.

(2) It is enough to show that $\mathbb{Q} = \mathbb{Q}_{\omega^2}^{\omega_1}$ is not σ -finite-cc. Towards a contradiction, suppose $\mathbb{Q} = \bigsqcup_{n < \omega} W_n$ where no W_n has an infinite antichain. Choose $\langle A_n : n < \omega \rangle$ as follows.

- (a) $A_0 \subseteq W_0$ is a maximal antichain of conditions p such that $\max(\operatorname{dom}(p)) = \gamma_p$ exists and $p(\gamma_p) = 1$. Define $\gamma_0 = \max(\{\gamma_p : p \in A_0\})$.
- (b) $A_{n+1} \subseteq W_{n+1}$ is a maximal antichain of conditions $p \in W_{n+1}$ such that $\max(\operatorname{dom}(p)) = \gamma_p$ exists, $\gamma_p > \gamma_n$ and $p(\gamma_p) = 1$. If $A_{n+1} \neq \emptyset$, define $\gamma_{n+1} = \max(\{\gamma_p : p \in A_{n+1}\})$. Otherwise, $\gamma_{n+1} = \gamma_n$.

Put $A = \bigcup_{n < \omega} A_n$ and $\gamma = \sup(\{\gamma_n : n < \omega\})$. Fix $\gamma_* \in (\gamma, \omega_1)$. Let p_* be defined by dom $(p_*) = \{\gamma_p : p \in A\} \cup \{\gamma_*\}$ and for every $\xi \in \text{dom}(p_*), p(\xi) = 1$ iff $\xi = \gamma_*$. Note that $\text{otp}(\text{dom}(p)) \le \omega + 1 < \omega^2$ and hence $p_* \in \mathbb{Q}$. Choose $n < \omega$ such that $p_* \in W_n$. But now $A_n \cup \{p_*\} \subseteq W_n$ is an antichain which contradicts the maximality of A_n .

Theorem 4.3. Suppose $\omega_1 \leq \kappa \leq \lambda$, \mathcal{I} is an ω_1 -saturated uniform ideal on λ and $\mathsf{add}(\mathcal{I}) = \kappa$. Let \mathbb{P}_{κ} be as in Definition 4.1. Let \mathcal{J} be the ideal generated by \mathcal{I} in $V^{\mathbb{P}_{\kappa}}$. Then there exists $\mathcal{A} \subseteq \mathcal{J}^+$ such that $|\mathcal{A}| = \omega_1$ and there is no countable set that meets every member of \mathcal{A} . Hence $V^{\mathbb{P}_{\kappa}} \models \mathcal{J}$ is an ω_1 -saturated κ -complete uniform ideal on λ which is not supersaturated.

Proof. As \mathbb{P}_{κ} is ccc, it is easy to see that in $V^{\mathbb{P}_{\kappa}}$, \mathcal{J} is an ω_1 -saturated κ -complete uniform ideal on λ . So it suffices to show that in $V^{\mathbb{P}_{\kappa}}$, there exists $\mathcal{A} \subseteq \mathcal{J}^+$ such that $|\mathcal{A}| = \omega_1$ and there is no countable set that meets every member of \mathcal{A} .

Since $\operatorname{add}(\mathcal{I}) = \kappa$, we can fix $Y \in \mathcal{I}^+$ and a partition $Y = \bigsqcup_{\alpha < \kappa} W_\alpha$ such that for each $\Gamma \in [\kappa]^{<\kappa}$, $\bigcup_{\alpha \in \Gamma} W_\alpha \in \mathcal{I}$. Let G be \mathbb{P}_{κ} -generic over V. Let $G_{\delta} = \{p(\delta) : p \in G\}$. So G_{δ} is \mathbb{Q}_{δ} -generic over V. Define $\mathring{A}_{\delta} \in V^{\mathbb{P}_{\kappa}} \cap \mathcal{P}(\lambda)$ by

$$\gamma \in \mathring{A}_{\delta} \iff (\exists p \in G)(p(\delta)(\alpha) = 1 \land \gamma \in W_{\alpha})$$

Suppose $Y \in \mathcal{I}$ and $p \in \mathbb{P}_{\kappa}$ with $\delta \in \operatorname{dom}(p)$. Choose $\alpha < \kappa$ such that $W_{\alpha} \setminus Y \neq \emptyset$ and $\alpha \notin \operatorname{dom}(p(\delta))$. Let $q \leq p$ be such that $q(\delta)(\alpha) = 1$. Then $q \Vdash_{\mathbb{P}_{\kappa}} \mathring{A}_{\delta} \setminus Y \neq \emptyset$. Hence $\Vdash_{\mathbb{P}_{\kappa}} \mathring{A}_{\delta} \in \mathcal{J}^+$.

Towards a contradiction suppose that in $V^{\mathbb{P}_{\kappa}}$, there is a countable $X \subseteq \lambda$ that meets each \mathring{A}_{δ} . Since \mathbb{P} satisfies ccc, we can assume that $X \in V$. Fix $p \in \mathbb{P}_{\kappa}$ such that $p \Vdash_{\mathbb{P}} (\forall \delta)(X \cap \mathring{A}_{\delta} \neq \emptyset)$. Put $W = \{\alpha < \kappa : W_{\alpha} \cap X \neq \emptyset\}$. So $W \subseteq \kappa$ is countable. Choose $\delta \in \omega_1 \setminus \operatorname{dom}(p)$ indecomposable such that $\delta > \operatorname{otp}(W)$. Define $q \in \mathbb{P}_{\kappa}$ by $\operatorname{dom}(q) = \operatorname{dom}(p) \cup \{\delta\}, q \upharpoonright \operatorname{dom}(p) = p$ and $q(\delta) \in \mathbb{Q}_{\delta}$ is constantly zero on W. Then $q \leq p$ and $q \Vdash_{\mathbb{P}_{\kappa}} X \cap \mathring{A}_{\delta} = \emptyset$: Contradiction. It follows that $\mathcal{A} = \{A_{\delta} : \delta < \omega_1, \delta \text{ indecomposable}\}$ is as required. \Box

Definition 4.4. Let $\langle (\mathbb{S}_i, \mathbb{R}_j) : i \leq \kappa^+, j < \kappa \rangle$ be the finite support iteration defined by

(a) \mathbb{S}_0 is the trivial forcing.

(b) For each $i < \kappa^+$, $V^{\mathbb{S}_i} \models \mathbb{R}_i = \mathbb{P}_{\kappa}$.

The next theorem shows how to kill all atomless supersaturated ideals.

Theorem 4.5. Suppose $V \models \text{``c} = \omega_1$ and κ is the least measurable cardinal with a witnessing normal prime ideal \mathcal{I} ''. Put $\mathbb{S} = \mathbb{S}_{\kappa^+}$. Then the following hold in $V^{\mathbb{S}}$.

- (a) $\mathfrak{c} = \kappa^+$ and the ideal generated by \mathcal{I} is a normal ω_1 -saturated ideal on κ .
- (b) Whenever \mathcal{J} is a supersaturated ideal on a set X, there is a countable partition \mathcal{F} of X such that for each $A \in \mathcal{F}$, $\mathcal{J} \upharpoonright A$ is a prime ideal. In particular, there is no supersaturated ideal on any cardinal $\leq \mathfrak{c}$.

Fact 4.6. Suppose \mathcal{I}_1 , \mathcal{I}_2 are ω_1 -saturated σ -ideals on X and $\mathcal{I}_1 \subseteq \mathcal{I}_2$. Then there is a partition $X = A \sqcup B$ such that $A \in \mathcal{I}_2$ and $\mathcal{I}_2 \upharpoonright B = \mathcal{I}_1 \upharpoonright B$.

Proof. Take A to be the union of a maximal family of pairwise disjoint sets in $\mathcal{I}_2 \setminus \mathcal{I}_1$.

The following lemma will be used in the proofs of Theorems 4.5 and 4.8(d). Recall that an ideal \mathcal{J} is nowhere prime iff every \mathcal{J} -positive set can be partitioned into two \mathcal{J} -positive subsets.

Lemma 4.7. Suppose \mathcal{J} is a nowhere prime supersaturated ideal on X and $\mu = add(\mathcal{J})$. Then $\mu \leq \mathfrak{c}$ and there exists a μ -additive supersaturated ideal on μ .

Proof. Towards a contradiction, suppose $\mu > \mathfrak{c}$. Construct a tree $\langle A_{\sigma} : \sigma \in 2^{<\omega_1} \rangle$ of subsets of X as follows.

- (i) $A_{\emptyset} = X$.
- (ii) If $A_{\sigma} \in \mathcal{J}^+$, then $\{A_{\sigma 0}, A_{\sigma 1}\}$ is a partition of A_{σ} into two \mathcal{J} -positive sets. This is possible since \mathcal{J} is nowhere prime.
- (iii) If $A_{\sigma} \in \mathcal{J}$, then $A_{\sigma 0} = A_{\sigma 1} = A_{\sigma}$.
- (iv) If $\alpha < \omega_1$ is limit and $\sigma \in 2^{\alpha}$, then $A_{\sigma} = \bigcap \{A_{\sigma \upharpoonright \beta} : \beta < \alpha \}$.

Put $\mathcal{F} = \{A_{\sigma} : \sigma \in 2^{<\omega_1} \text{ and } A_{\sigma} \in \mathcal{J}\}$. We claim that $X = \bigcup \mathcal{F}$. Suppose not and fix $y \in X \setminus \bigcup \mathcal{F}$. Now observe that $\{A_{\sigma k} : \sigma \in 2^{<\omega_1} \land k < 2 \land y \in (A_{\sigma} \setminus A_{\sigma k})\}$ is an uncountable family of pairwise disjoint \mathcal{J} -positive sets which contradicts the fact that \mathcal{J} is ω_1 -saturated. So $X = \bigcup \mathcal{F}$. But since $|\mathcal{F}| \leq |2^{<\omega_1}| = \mathfrak{c}$, this contradicts the fact that $\operatorname{add}(\mathcal{J}) = \mu > \mathfrak{c}$. Hence $\mu \leq \mathfrak{c}$.

Since $\operatorname{add}(\mathcal{J}) = \mu$, there are $Y \in \mathcal{J}^+$ and a partition $Y = \bigsqcup_{\alpha < \mu} W_{\alpha}$ such that for every $\Gamma \in [\mu]^{<\mu}, \bigcup_{\alpha \in \Gamma} W_{\alpha} \in \mathcal{J}$. Define

$$\mathcal{K} = \{ \Gamma \subseteq \mu : \bigcup_{\alpha \in \Gamma} W_{\alpha} \in \mathcal{J} \}$$

Then \mathcal{K} is a μ -additive ω_1 -saturated ideal on μ . So μ is weakly inaccessible. We claim that \mathcal{K} must also be supersaturated. To see this, suppose $\mathcal{A} \subseteq \mathcal{K}^+$ and $|\mathcal{A}| < \mu$. For each $A \in \mathcal{A}$, define $Y_A = \bigsqcup_{\alpha \in A} W_{\alpha}$. Then $\{Y_A : A \in \mathcal{A}\} \subseteq \mathcal{J}^+$.

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Since \mathcal{J} is supersaturated, we can choose a countable $T \subseteq Y$ that meets Y_A for every $A \in \mathcal{A}$. Let $B = \{\alpha < \mu : T \cap W_\alpha \neq \emptyset\}$. Then $B \subseteq \mu$ is countable (as W_α 's are pairwise disjoint) and it meets every $A \in \mathcal{A}$. Hence \mathcal{K} is a μ -additive supersaturated ideal on μ .

Proof of Theorem 4.5: Clause (a) is easy to check. Let us prove Clause (b). Suppose \mathcal{J} is a supersaturated ideal on X. Put $\mu = \mathsf{add}(\mathcal{J})$. We claim that it suffices to show that $V^{\mathbb{S}} \models \mu > \mathfrak{c}$. First note that, by Lemma 4.7, this would imply that for every $Y \in \mathcal{J}^+$, there exists \mathcal{J} -positive $Z \subseteq Y$ such that $\mathcal{J} \upharpoonright Z$ is a prime ideal. Hence by ω_1 -saturation of \mathcal{J} , we can find a countable partition of X into \mathcal{J} -positive sets such that the restriction of \mathcal{J} to each one of them is a prime ideal.

So towards a contradiction, assume $V^{\mathbb{S}} \models \mu \leq \mathfrak{c}$. Fix $Y \in \mathcal{J}^+$ such that for every \mathcal{J} -positive $Z \subseteq Y$, $\mathsf{add}(\mathcal{J} \upharpoonright Z) = \mu$. Since $\mu \leq \mathfrak{c}$, it follows that $\mathcal{J} \upharpoonright Y$ is a nowhere prime supersaturated ideal. Using Lemma 4.7 again, we can get a μ -additive supersaturated ideal \mathcal{K} on μ . Let us assume that the trivial condition in \mathbb{S} forces all of this about \mathcal{K} .

Since $V^{\mathbb{S}} \models ``\mu \leq \mathfrak{c} = \kappa^+$ and μ is weakly inaccessible", we must have $\mu \leq \kappa$. We consider two cases.

Case $\mu < \kappa$: In V, define $\mathcal{I}' = \{X \subseteq \mu : \mathbb{1}_{\mathbb{S}} \Vdash X \in \mathcal{K}\}$. Since S is ccc, $V \models \mathcal{I}'$ is a μ -additive ω_1 -saturated ideal on μ . As $V \models \mu > \omega_1 = \mathfrak{c}$, μ is measurable in V. Since κ is the least measurable cardinal in $V, \mu \geq \kappa$: Contradiction.

Case $\mu = \kappa$: In V, define $\mathcal{I}' = \{X \subseteq \kappa : \mathbb{1}_{\mathbb{S}} \Vdash X \in \mathcal{K}\}$. Since $V \models \kappa > \mathfrak{c} = \omega_1$, we must have $V \models \mathcal{I}'$ is a κ -additive prime ideal on κ . Let \mathcal{K}' be the ideal generated by \mathcal{I}' in $V^{\mathbb{S}}$. Then $V^{\mathbb{S}} \models \mathcal{K}' \subseteq \mathcal{K}$ are ω_1 -saturated κ -additive ideals on κ . Using Fact 4.6, fix $B \in \mathcal{K}^+$ such that $\mathcal{K}' \upharpoonright B = \mathcal{K} \upharpoonright B$.

Choose $\gamma < \kappa^+$ such that $\mathring{B} \in V^{\mathbb{S}_{\gamma}}$. Let \mathcal{K}'' be the ideal generated by \mathcal{I}' in $V^{\mathbb{S}_{\gamma}}$. By Theorem 4.3, it follows that in $V^{\mathbb{S}_{\gamma+1}}$, the ideal generated by $\mathcal{K}'' \upharpoonright B$ is not supersaturated. Now observe that $\mathcal{K} \upharpoonright B = \mathcal{K}' \upharpoonright B$ is the ideal generated by $\mathcal{K}'' \upharpoonright B$ in $V^{\mathbb{S}}$. It follows that \mathcal{K} is not a supersaturated ideal: Contradiction. \Box

Using some results about separating families and supersaturated ideals from [2, 4], we can also get the following.

Theorem 4.8. Suppose κ is a measurable cardinal with a witnessing normal prime ideal \mathcal{I} . Let \mathbb{P}_{κ} be the forcing in Definition 4.1. Then the following hold in $V^{\mathbb{P}_{\kappa}}$.

- (a) $\mathfrak{c} = \kappa$ and the ideal generated by \mathcal{I} is a normal ω_1 -saturated ideal on κ .
- (b) There is a family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ such that $|\mathcal{F}| = \omega_1$ and for every countable $X \subseteq \kappa$ and $\alpha \in \kappa \setminus X$, there exists $S \in \mathcal{F}$ such that $\alpha \in S$ and $S \cap X = \emptyset$.
- (c) The order dimension of Turing degrees is ω_1 .
- (d) There are no nowhere prime supersaturated ideals.

Proof. (a) Since $\mathbb{Q}_{\omega}^{\kappa}$ adds κ Cohen reals, $\mathfrak{c} \geq \kappa$. The other inequality follows by a name counting argument using the facts that \mathbb{P}_{κ} is a ccc forcing, $|\mathbb{P}_{\kappa}| = \kappa$ and $\kappa^{\omega} = \kappa$. That the ideal generated by \mathcal{I} is a normal ω_1 -saturated ideal on κ follows from the fact that \mathbb{P}_{κ} is ccc.

(b) For each indecomposable $\delta < \omega_1$, define

 $S_{\delta} = \{ \alpha < \kappa : (\exists p \in G_{\mathbb{P}_{\kappa}}) (\delta \in \mathsf{dom}(p) \land p(\delta)(\alpha) = 1) \}$

Let $\mathcal{F} = \{S_{\delta} : \delta < \omega_1 \text{ is indecomposable}\}$. Suppose $X \subseteq \kappa$ is countable and $\alpha \in \kappa \setminus X$. We'll find an $S_{\delta} \in \mathcal{F}$ such that $\alpha \in S_{\delta}$ and $X \cap S_{\delta} = \emptyset$. Since \mathbb{P}_{κ} is ccc, we can find a countable $Y \in V$ such that $X \subseteq Y \subseteq \kappa \setminus \{\alpha\}$. Now an easy density argument shows that the set

$$D_{\alpha,Y} = \{ p \in \mathbb{P}_{\kappa} : (\exists \delta \in \mathsf{dom}(p)) [p(\delta)(\alpha) = 1 \land (\forall \beta \in Y) (p(\delta)(\beta) = 0)] \}$$

is dense in \mathbb{P}_{κ} . So we can choose $p \in D_{\alpha,Y} \cap G_{\mathbb{P}_{\kappa}}$. Let δ witness that $p \in D_{\alpha,Y}$. Then it is clear that $\alpha \in S_{\delta}$ and $X \cap S_{\delta} \subseteq Y \cap S_{\delta} = \emptyset$.

(c) This follows from Theorem 3.9 in [2] and part (b) above.

(d) Suppose not. Then by Lemma 4.7, we can find some $\mu \leq \mathfrak{c} = \kappa$ and a μ -additive supersaturated ideal on μ . Let \mathcal{F} be as in part (b) above. Define $\mathcal{E} = \{S \cap \mu : S \in \mathcal{F}\}$. Then $|\mathcal{E}| = \omega_1$ and for every countable $X \subseteq \mu$ and $\alpha \in \mu \setminus X$, there exists $S \in \mathcal{E}$ such that $\alpha \in S$ and $S \cap X = \emptyset$. Now applying Lemma 4.2 in [4] gives us a contradiction.

We conclude with the following questions.

- (1) Suppose \mathcal{I}, \mathcal{J} are normal ideals on κ , \mathcal{I} is supersaturated and $\mathcal{P}(\kappa)/\mathcal{I}$ is isomorphic to $\mathcal{P}(\kappa)/\mathcal{J}$. Must \mathcal{J} be supersaturated?
- (2) Suppose κ is regular uncountable, \mathcal{I} is a κ -complete normal ideal on κ and $\mathcal{P}(\kappa)/\mathcal{I}$ is a Cohen algebra. Must \mathcal{I} be supersaturated?
- (3) Do σ -finite/bounded-cc forcings preserve supersaturation? What about Boolean algebras that admit a strictly positive finitely additive measure?

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