COMBINATORIAL PROPERTIES OF MAD FAMILIES

JÖRG BRENDLE, OSVALDO GUZMÁN, MICHAEL HRUŠÁK, AND DILIP RAGHAVAN

Abstract. We study some strong combinatorial properties of MAD families. An ideal $\mathcal{I}$ is Shelah-Steprāns if for every set $X \subseteq [\omega]^{<\omega}$ there is an element of $\mathcal{I}$ that either intersects every set in X or contains infinitely many members of it. We prove that a Borel ideal is Shelah-Steprāns if and only if it is Katětov above the ideal fin$\times$fin. We prove that Shelah-Steprāns MAD families have strong indestructibility properties (in particular, they are both Cohen and random indestructible). We also consider some other strong combinatorial properties of MAD families. It is proved that it is consistent to have $\text{non}(\mathcal{M}) = \aleph_1$ and no Shelah-Steprāns families of size $\aleph_1$. We develop a general machinery for producing models of set theory with $\text{non}(\mathcal{M}) = \aleph_1$ in order to prove this result.

1. Introduction and preliminaries

In [23] Katětov introduced a preorder on ideals. The Katětov order is a very powerful tool for studying ideals over countable sets. For the convenience of the reader, we will now recall the definition of this order: Let X and Y be two countable sets, $\mathcal{I}, \mathcal{J}$ ideals on X and Y respectively and $f : Y \longrightarrow X$. We say f is a *Katětov-morphism from* $(Y, \mathcal{J})$ *to* $(X, \mathcal{I})$ if $f^{-1}(A) \in \mathcal{J}$ for every $A \in \mathcal{I}$. We say $\mathcal{I} \leq_K \mathcal{J}$ ($\mathcal{I}$ is *Katětov smaller than* $\mathcal{J}$ or $\mathcal{J}$ is *Katětov above* $\mathcal{I}$) if there is a Katětov-morphism from $(Y, \mathcal{J})$ to $(X, \mathcal{I})$. We say $\mathcal{I} \simeq_K \mathcal{J}$ (*$\mathcal{I}$ is Katětov equivalent to $\mathcal{J}$*) if $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \leq_K \mathcal{I}$. The *Katětov-Blass order* (denoted by $\leq_{KB}$) is defined in the same way as the Katětov order, but with the additional demand that the function f must be finite to one. The Katětov order has been applied successfully in classifying non definable objects such as ultrafilters and MAD families. Just to mention a couple of examples, an ultrafilter $\mathcal{U}$ is a P-point if and only if the dual ideal $\mathcal{U}^*$ is not Katětov above the ideal fin$\times$fin and $\mathcal{U}$ is a Ramsey ultrafilter if and only if $\mathcal{U}^*$ is not Katětov above the ideal $\mathcal{ED}$. In fact, most of the usual properties of ultrafilters can be characterized with the Katětov order. In the case of ultrafilters, the upward cones of definable ideals in the Katětov order play a fundamental role, while the downward cones of definable ideals are very important in the study of MAD families. The reader may consult [19] for a survey on the Katětov order on Borel ideals.

Date: November 7, 2022.

2010 *Mathematics Subject Classification.* 03E17, 03E35, 03E05, 03E55.

Key words and phrases. cardinal invariants, almost disjoint families, non-meager sets.

The research of the first author was partially supported by Grants-in-Aid for Scientific Research (C) 15K04977 and (C) 18K03398, Japan Society for the Promotion of Science. Part of his work was done while he was visiting the Institute for Mathematical Sciences, National University of Singapore, during the program on *Aspects of Computation* (21 Aug – 15 Sep 2017); the visit was supported by the Institute.

The second author was supported by by CONACyT scholarship 420090 and by NSERC of Canada.

The third author was supported by a PAPIIT grants IN 102311, IN 104220 and CONACyT grant A1-S-16164.

This paper was completed when the fourth author was a Fields Research Fellow. The fourth author thanks the Fields Institute for its kind hospitality.

Recall that a family $\mathcal{A} \subseteq [\omega]^{\omega}$ is *almost disjoint (AD)* if the intersection of any two different elements of $\mathcal{A}$ is finite and a *MAD family* is a maximal almost disjoint family. If $\mathcal{A}$ is an AD family, we denote by $\mathcal{I}(\mathcal{A})$ the ideal generated by $\mathcal{A}$ and all finite sets of ω. This paper is part of a larger project (initiated in [20]) whose goal is to study and classify (the ideals generated by) MAD families using the Katětov order.

If $\mathcal{A}$ is a MAD family, we say that a forcing notion $\mathbb{P}$ *destroys* $\mathcal{A}$ if $\mathcal{A}$ is no longer maximal after forcing with $\mathbb{P}$. The destructibility of MAD families has been extensively studied (see [8], [24] and [22]). Nevertheless, many fundamental questions are still open. For example, the following problems still remain unsolved:

Problem 1.1 (Steprāns)**.** *Is there a Cohen indestructible MAD family?*

Problem 1.2 (Hrušák)**.** *Is there a Sacks indestructible MAD family?*

The answer to both questions is positive under many additional axioms, but it is currently unknown if it is possible to build such families on the basis of ZFC alone. It is easy to see that there is a MAD family that is destroyed by every forcing adding a new real, so the main interest is to construct MAD families that are indestructible under certain forcings adding reals. It is known that every forcing adding a dominating real will destroy every ground model MAD family.

In this article, we study some strong combinatorial properties of MAD families: Shelah-Steprāns, strongly tight and raving (see the next sections for the definitions). We will prove that Shelah-Steprāns MAD families have very strong indestructibility properties, in fact, they are indestructible by most definable forcings that do not add dominating reals (see Proposition 4.13). The notion of strongly tight is a strengthening of Cohen indestructibility, yet they may be random destructible. Raving is a strengthening of both Shelah-Steprāns and strong tightness.

If $\mathcal{A}$ is an AD family, we denote by $\mathcal{A}^{\perp}$ the set of all $B \subseteq \omega$ that are almost disjoint with every element of $\mathcal{A}$. If $\mathcal{I}$ is an ideal, we denote by $\mathcal{I}^{+}$ the family of subsets of ω that are not in $\mathcal{I}$. $\mathcal{I}^{*} = \{\omega \setminus A : A \in \mathcal{I}\}$ is the *dual filter* of $\mathcal{I}$. We say that a forcing $\mathbb{P}$ *destroys (or diagonalizes)* $\mathcal{I}$ if $\mathcal{I}$ is no longer tall after forcing with $\mathbb{P}$[1]. It is easy to see that a forcing $\mathbb{P}$ destroys a MAD family $\mathcal{A}$ if and only if it destroys $\mathcal{I}(\mathcal{A})$. The Katětov order is a fundamental tool for studying the indestructibility of MAD families and ideals. The following notion is needed in order to connect the Katětov order and the notion of indestructibility:

Definition 1.3. *For every $a \subseteq \omega^{<\omega}$ we define $\pi(a) = \{f \in \omega^{\omega} \mid \exists^{\infty} n\, (f \restriction n \in a)\}$. If $\mathcal{I}$ is a σ-ideal in ω^{ω} (or 2^{ω}) we define* the trace ideal $tr(\mathcal{I})$ of $\mathcal{I}$ *(which will be an ideal in $\omega^{<\omega}$ or $2^{<\omega}$) such that $a \in tr(\mathcal{I})$ if and only if $\pi(a) \in \mathcal{I}$.*

Note that if $a \subseteq \omega^{<\omega}$ then $\pi(a)$ is a G_{δ} set (furthermore, every G_{δ} set is of this form). While both $tr(\mathcal{M})$ and $tr(\mathcal{N})$ are Borel (where $\mathcal{M}$ denotes the ideal of meager sets of ω^{ω} and $\mathcal{N}$ is the ideal of all null sets), in general, the trace ideals are not Borel (see [22] for more information). The relevance of the trace ideals in the study of destructibility is the following result of Hrušák and Zapletal (see also [8]):

Proposition 1.4 ([22])**.** *Let $\mathcal{I}$ be a σ-ideal in ω^{ω} such that $\mathbb{P}_{\mathcal{I}} = Borel(\omega^{\omega})/\mathcal{I}$ is proper and has the continuous reading of names*[2]*. If $\mathcal{J}$ is an ideal on ω, then the following are equivalent:*

(1) There is a condition $B \in \mathbb{P}_{\mathcal{I}}$ such that B forces that $\mathcal{J}$ is not tall.
(2) There is $a \in tr(\mathcal{I})^{+}$ such that $\mathcal{J} \leq_K tr(\mathcal{I}) \restriction a$.

[1] An ideal $\mathcal{I}$ is *tall* if for every $X \in [\omega]^{\omega}$ there is $A \in \mathcal{I}$ such that $A \cap X$ is infinite.

[2] see [45] for the definition of continuous reading of names.

Usually, we assume ideals are proper and contain all finite sets. However, there is an exception to this convention, which we will point out (see Subsection 4.2). For every $n \in \omega$ we define $C_n = \{(n,m) \mid m \in \omega\}$ and if $f : \omega \longrightarrow \omega$ let $D(f) = \{(n,m) \mid m \leq f(n)\}$. The ideal $\mathsf{fin}\times\mathsf{fin}$ is the ideal on $\omega \times \omega$ generated by $\{C_n \mid n \in \omega\} \cup \{D(f) \mid f \in \omega^\omega\}$ and $\emptyset\times\mathsf{fin}$ is the ideal on $\omega \times \omega$ generated by $\{D(f) \mid f \in \omega^\omega\}$. Note that $\mathsf{fin}\times\mathsf{fin}$ is a tall ideal while $\emptyset\times\mathsf{fin}$ is not. It is well known that a forcing destroys $\mathsf{fin}\times\mathsf{fin}$ if and only if it adds a dominating real. By nwd we denote the ideal of all nowhere dense subsets of the rational numbers. The *density zero* ideal is defined as $\mathcal{Z} = \{A \subseteq \omega \mid \lim \frac{|A \cap 2^n|}{2^n} = 0\}$. It is well known that $\mathcal{Z}$ is an analytic P-ideal. The *summable ideal* is defined as $\mathcal{J}_{1/n} = \{A \subseteq \omega \mid \sum\limits_{n \in A} \frac{1}{n+1} < \omega\}$. It is well known and easy to see that this is an F_σ-ideal. In fact, F_σ-ideals and analytic P-ideals have a canonical representation:

Definition 1.5. *We say $\varphi : \wp(\omega) \longrightarrow \mathbb{R} \cup \{\infty\}$ is a* lower semicontinuous submeasure *if the following hold:*

(1) $\varphi(\emptyset) = 0$.
(2) $\varphi(A) \leq \varphi(B)$ *whenever* $A \subseteq B$.
(3) $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ *for every* $A, B \subseteq X$.
(4) (lower semicontinuity) if $A \subseteq \omega$ *then* $\varphi(A) = \sup\{\varphi(A \cap n) \mid n \in \omega\}$.

Given a lower semicontinuous submeasure φ we define $\mathsf{fin}(\varphi)$ as those subsets of ω with finite submeasure and $\mathsf{Exh}(\varphi) = \{A \subseteq \omega \mid \lim(\varphi(A \setminus n)) = 0\}$. The following are two fundamental results:

Proposition 1.6. *Let $\mathcal{I}$ be an ideal in ω.*

(1) (Mazur [28]) *$\mathcal{I}$ is an F_σ-ideal if and only if there is a lower semicontinuous submeasure φ such that $\mathcal{I} = \mathsf{fin}(\varphi)$.*
(2) (Solecki [43]) *$\mathcal{I}$ is an analytic P-ideal if and only if there is a lower semicontinuous submeasure φ such that $\mathcal{I} = \mathsf{Exh}(\varphi)$ (in particular, every analytic P-ideal is $F_{\sigma\delta}$).*

For the definition of the cardinal invariants used in this paper, the reader may consult [3].

2. Notation

We will use the following notational conventions for iterated forcing in Sections 6, 7.1, and 7.2. Let $\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ and $\langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle$ be posets and let $\pi : \mathbb{Q} \to \mathbb{P}$ be a projection. If $G \subseteq \mathbb{P}$ is a $(\mathbf{V}, \mathbb{P})$-generic filter, then in $\mathbf{V}[G]$ we define the poset $\mathbb{Q}/G = \{q \in \mathbb{Q} : \pi(q) \in G\}$ ordered by $\leq_{\mathbb{Q}}$. In $\mathbf{V}$, we let $\mathbb{Q}/\mathring{G}$ be a full $\mathbb{P}$-name for $\mathbb{Q}/G$.

Suppose $\langle \mathbb{P}_\alpha ; \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ is an iteration. If G_γ is a $(\mathbf{V}, \mathbb{P}_\gamma)$-generic filter, then for any $\alpha \leq \gamma$, G_α denotes $\{p \restriction \alpha : p \in G_\gamma\}$, and it is a $(\mathbf{V}, \mathbb{P}_\alpha)$-generic filter. Furthermore, if $\mathring{x}$ is any $\mathbb{P}_\alpha$-name, then $\mathring{x}$ is also a $\mathbb{P}_\beta$-name for every $\alpha \leq \beta \leq \gamma$ and $\mathring{x}[G_\alpha] = \mathring{x}[G_\beta]$.

If $\langle \mathbb{P}_\alpha ; \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ is an iteration, then for each $\alpha \leq \gamma$ the map $\pi_{\gamma\alpha} : \mathbb{P}_\gamma \to \mathbb{P}_\alpha$ given by $\pi_{\gamma\alpha}(p) = p \restriction \alpha$ is a projection. Therefore, if $\alpha \leq \gamma$ and if $G_\gamma \subseteq \mathbb{P}_\gamma$ is a $(\mathbf{V}, \mathbb{P}_\gamma)$-generic filter, then there is a $(\mathbf{V}[G_\alpha], \mathbb{P}_\gamma/G_\alpha)$-generic filter H so that in $\mathbf{V}[G_\gamma]$, $G_\gamma = G_\alpha * H$ holds. In fact, this H is equal to G_γ.

Suppose $\langle \mathbb{P}_\alpha ; \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ is an iteration and let $\alpha \leq \gamma$. We may think of any $\mathbb{P}_\gamma$-name as a $\mathbb{P}_\alpha$-name for a $\mathbb{P}_\gamma/\mathring{G}_\alpha$-name. Thus, given a $\mathbb{P}_\gamma$ name $\mathring{x}$, we use $\mathring{x}[\mathring{G}_\alpha]$ to denote a canonical $\mathbb{P}_\alpha$-name for a $\mathbb{P}_\gamma/\mathring{G}_\alpha$-name representing $\mathring{x}$. If G_α is a $(\mathbf{V}, \mathbb{P}_\alpha)$-generic filter, we will write $\mathring{x}[G_\alpha]$ to denote the evaluation of $\mathring{x}[\mathring{G}_\alpha]$ by

G_α. Therefore, if G_γ is a $(\mathbf{V}, \mathbb{P}_\gamma)$-generic filter, then in $\mathbf{V}[G_\gamma]$, $\mathring{x}[G_\gamma] = \mathring{x}[G_\alpha][G_\gamma]$ holds.

3. Shelah-Steprāns ideals

Let $\mathcal{I}$ be an ideal in ω. By $\mathcal{I}^{<\omega}$ we denote the ideal of subsets X of $[\omega]^{<\omega} \setminus \{\emptyset\}$ for which there exists $A \in \mathcal{I}$ such that $s \cap A \neq \emptyset$ for all $s \in X$. Thus $(\mathcal{I}^{<\omega})^+$ is the set of all $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ such that for every $A \in \mathcal{I}$ there is $s \in X$ such that $s \cap A = \emptyset$. The following notion will play a fundamental role in this paper:

Definition 3.1. *An ideal $\mathcal{I}$ is called* Shelah-Steprāns *if for every $X \in (\mathcal{I}^{<\omega})^+$ there is $Y \in [X]^\omega$ such that $\bigcup Y \in \mathcal{I}$.*

In other words, an ideal $\mathcal{I}$ is Shelah-Steprāns if for every $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ either there is $A \in \mathcal{I}$ such that $s \cap A \neq \emptyset$ for every $s \in X$ or there is $B \in \mathcal{I}$ that contains infinitely many elements of X. This notion, introduced by Raghavan in [36] for almost disjoint families, is connected to the notion of "strongly separable" introduced by Shelah and Steprāns in [42].

Lemma 3.2. *Every non-meager ideal is Shelah-Steprāns.*

Proof. Let $\mathcal{I}$ be a non-meager ideal and $X \in (\mathcal{I}^{<\omega})^+$. Note that since $X \in (\mathcal{I}^{<\omega})^+$ (and $\mathcal{I}$ contains every finite set) for every $n \in \omega$ there is $s \in X$ such that $s \cap n = \emptyset$. In this way we can find $Z = \{s_n \mid n \in \omega\} \subseteq X$ such that if $n \neq m$ then $s_n \cap s_m = \emptyset$. We then define $M = \{A \subseteq \omega \mid \forall^\infty n\, (s_n \nsubseteq A)\}$ which is clearly a meager set and thus there must be $A \in \mathcal{I}$ such that $A \notin M$. Hence there is $Y \in [X]^\omega$ such that $\bigcup Y \subseteq A \in \mathcal{I}$. ⊣

Nevertheless, there are meager ideals that are also Shelah-Steprāns as the following result shows:

Lemma 3.3. *$\mathsf{fin}\times\mathsf{fin}$ is Shelah-Steprāns.*

Proof. It is easy to see that if $X \in ((\mathsf{fin}\times\mathsf{fin})^{<\omega})^+$ then there must be infinitely many elements of X that are below the graph of a function, so there must be $Y \in [X]^\omega$ such that $\bigcup Y \in \mathsf{fin}\times\mathsf{fin}$. ⊣

We now show that the property of being Shelah-Steprāns is upward closed in the Katětov order:

Lemma 3.4. *Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals on ω. If the ideal $\mathcal{I}$ is Shelah-Steprāns and $\mathcal{I} \leq_K \mathcal{J}$ then $\mathcal{J}$ is also Shelah-Steprāns.*

Proof. Let $f : \omega \longrightarrow \omega$ be a Katětov-morphism from $(\omega, \mathcal{J})$ to $(\omega, \mathcal{I})$. Letting $X \in (\mathcal{J}^{<\omega})^+$ we must find $Y \in [X]^\omega$ such that $\bigcup Y \in \mathcal{J}$. Defining $X_1 = \{f[s] \mid s \in X\}$, we will first argue that $X_1 \in (\mathcal{I}^{<\omega})^+$. To prove this fact, let $A \in \mathcal{I}$. Since f is a Katětov-morphism, $f^{-1}(A) \in \mathcal{J}$ so there is $s \in X$ for which $s \cap f^{-1}(A) = \emptyset$ and then $f[s] \cap A = \emptyset$. Since $\mathcal{I}$ is Shelah-Steprāns, there is $Y_1 \in [X_1]^\omega$ such that $\bigcup Y_1 \in \mathcal{I}$. Finally if $Y \in [X]^\omega$ is such that $Y_1 = \{f[s] \mid s \in Y\}$ then $\bigcup Y \in \mathcal{J}$. ⊣

We will need the following game designed by Claude Laflamme: Letting $\mathcal{I}$ be an ideal on ω, define the game $\mathcal{L}(\mathcal{I})$ between players I and II as follows:

I	...	A_n		...	
II	...		s_n	...	$\bigcup s_n \in \mathcal{I}^+$

At round $n \in \omega$ player I plays $A_n \in \mathcal{I}$ and II responds with $s_n \in [\omega \setminus A_n]^{<\omega}$. The player II wins in case $\bigcup s_n \in \mathcal{I}^+$. The following is a result of Laflamme.

Proposition 3.5 (Laflamme [26]). *Let $\mathcal{I}$ be an ideal on ω.*

(1) *The following are equivalent:*
   (a) I *has a winning strategy in* $\mathcal{L}(\mathcal{I})$.
   (b) *For every* $\{F_s \mid s \in \omega^{<\omega}\} \subseteq \mathcal{I}^*$, *there is an increasing function* $f \in \omega^\omega$ *such that* $\bigcup_{n\in\omega} (F_{f\restriction n} \cap f(n)) \in \mathcal{I}^+$.
   (c) *Every countable subset of* $\mathcal{I}^*$ *has a pseudointersection in* $\mathcal{I}^+$.
   (d) $\mathsf{fin}\times\mathsf{fin} \leq_{\mathsf{K}} \mathcal{I}$.
(2) *The following are equivalent:*
   (a) II *has a winning strategy in* $\mathcal{L}(\mathcal{I})$.
   (b) *There is* $\{X_n \mid n \in \omega\} \subseteq (\mathcal{I}^{<\omega})^+$ *such that for every* $A \in \mathcal{I}$ *there is* $n \in \omega$ *such that* $A \cap X_n = \emptyset$.

Proof. Everything in the Proposition is either contained in [26] or is trivial, with the exception of (c) implies (b) of 1, which is only mentioned in [26] without proof. We will provide a brief sketch of the proof of this implication.

Let $\{F_s \mid s \in \omega^{<\omega}\} \subseteq \mathcal{I}^*$. Without lost of generality, we may assume that for every $s, t \in \omega^{<\omega}$, if $t \subseteq s$, then $F_s \subseteq F_t$. For every $n \in \omega$, define:

$$H_n = \bigcap\{F_s \mid s \in n^{\leq n} \wedge \forall i \in dom(s)\,(s(i) \leq n)\}$$

Note that $H_n \in \mathcal{I}^*$ for every $n \in \omega$. By point (c), there is $X \in \mathcal{I}^+$ a pseudointersection of $\{H_n \mid n \in \omega\}$. We now find a sequence $\langle n_i \rangle_{i\in\omega}$ such that for every $i \in \omega$, the following holds:

(1) $n_i < n_{i+1}$.
(2) $X \setminus n_{i+1} \subseteq H_{n_i}$.

We now define the following sets:

$$Y_0 = X \cap (\bigcup\{[n_i, n_{i+1}) \mid i \text{ is even}\})$$
$$Y_1 = X \cap (\bigcup\{[n_i, n_{i+1}) \mid i \text{ is odd}\})$$

Since $X \in \mathcal{I}^+$, there is $i \in 2$ such that $Y_i \in \mathcal{I}^+$. If $Y_0 \in \mathcal{I}^+$, define $f \in \omega^\omega$ where $f(m) = n_{2m+1}$ and if $Y_1 \in \mathcal{I}^+$, define f by $f(m) = n_{2(m+1)}$. In either case, it is easy to see that $\bigcup_{m\in\omega} (F_{f\restriction m} \cap f(m)) \in \mathcal{I}^+$. $\dashv$

If $s_0, ..., s_n$ are finite non-empty subsets of ω, we say $a = \{k_0, ..., k_n\} \in [\omega]^{<\omega}$ is a *selector* of $(s_0, ..., s_n)$ if $k_i \in s_i$ for every $i \leq n$.

Proposition 3.6. *If* $\mathcal{I}$ *is Shelah-Steprāns then* II *does not have a winning strategy in* $\mathcal{L}(\mathcal{I})$.

Proof. Let $\mathcal{I}$ be an ideal for which II has a winning strategy in $\mathcal{L}(\mathcal{I})$. We will prove that $\mathcal{I}$ is not Shelah-Steprāns. Let $\{X_n \mid n \in \omega\} \subseteq (\mathcal{I}^{<\omega})^+$ such that for every $A \in \mathcal{I}$ there is $n \in \omega$ such that A does not contain any element of X_n. For every $n \in \omega$ enumerate $X_n = \{t_n^i \mid i \in \omega\}$ and $\prod_{j<n} X_j = \{p_n^i \mid i < \omega\}$.

For every $n, m \in \omega$ and a selector $a \in [\omega]^{<\omega}$ of $(t_n^0, ..., t_n^m)$ we define $F_{(n,m,a)} = p_n^m(0) \cup ... \cup p_n^m(n-1) \cup a$ (recall $p_n^m \in \prod_{j<n} X_j$). Clearly each $F_{(n,m,a)}$ is a non-empty finite set. Let X be the collection of all the $F_{(n,m,a)}$; we will prove that X witnesses that $\mathcal{I}$ is not Shelah-Steprāns.

We will first prove that $X \in (\mathcal{I}^{<\omega})^+$. Letting $A \in \mathcal{I}$ we first find $n \in \omega$ such that A does not contain any element of X_n. Since each $X_j \in (\mathcal{I}^{<\omega})^+$ for every $j < \omega$ there is $m \in \omega$ such that A is disjoint with $p_n^m(0) \cup ... \cup p_n^m(n-1)$. Finally, by the assumption of X_n we can find a selector b of $\left(t_n^0, ..., t_n^m\right)$ such that $b \cap A = \emptyset$ and therefore $A \cap F_{(n,m,b)} = \emptyset$.

Letting $Y \in [X]^\omega$ we will show that $B = \bigcup Y \in \mathcal{I}^+$. There are two cases to consider: first assume there is $n \in \omega$ for which there are infinitely many (m, a) such that $F_{(n,m,a)} \in Y$. In this case, B intersects every element of X_n, hence $B \in \mathcal{I}^+$. Now assume that for every $n \in \omega$ there are only finitely many (m, a) such that $F_{(n,m,a)} \in Y$. In this case, there must be infinitely many $n \in \omega$ for which there is (m, a) such that $F_{(n,m,a)} \in Y$, hence B must contain (at least) one element of every X_k. We can then conclude that $B \in \mathcal{I}^+$. ⊣

As a consequence we obtain (the equivalence of items 2 and 3 was proved by Laczkovich and Recław in [25]; we include the proof for the convenience of the reader):

Corollary 3.7. *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent:*

(1) *$\mathcal{I}$ is not Shelah-Steprāns.*
(2) *The Player* II *has a winning strategy in $\mathcal{L}(\mathcal{I})$.*
(3) *There is an F_σ set $F \subseteq \wp(\omega)$ such that $\mathcal{I} \subseteq F$ and $\mathcal{I}^* \cap F = \emptyset$.*

Proof. By the previous result, we know that 2 implies 1. We first prove that 1 implies 3. Assume that $\mathcal{I}$ is not Shelah-Steprāns, so there is $X = \{s_n \mid n \in \omega\} \in (\mathcal{I}^{<\omega})^+$ such that $\bigcup Y \in \mathcal{I}^+$ for every $Y \in [X]^\omega$. We now define the set $F = \{W \subseteq \omega \mid \forall^\infty n\,(s_n \nsubseteq W)\}$. It is easy to see that F has the desired properties.

We finally prove that 3 implies 2. Assume there is an increasing sequence of closed sets $\langle C_n \mid n \in \omega\rangle$ such that $F = \bigcup\limits_{n\in\omega} C_n$ contains $\mathcal{I}$ and is disjoint from $\mathcal{I}^*$. We will now describe a winning strategy for Player II in $\mathcal{L}(\mathcal{I})$: In the first round, if Player I plays $A_0 \in \mathcal{I}$ then Player II finds an initial segment s_0 of $\omega \setminus A_0$ such that $\langle s_0\rangle = \{Z \mid s_0 \sqsubseteq Z\}$ is disjoint from C_0 (where $s_0 \sqsubseteq Z$ means that s_0 is an initial segment of Z). At round round $n+1$, if Player I plays $A_{n+1} \in \mathcal{I}$ then Player II finds s_{n+1} such that $t = \bigcup\limits_{i\leq n+1} s_i$ is an initial segment of $(\omega \setminus A_n) \cup \bigcup\limits_{j<n+1} s_j$ (we may assume $\bigcup\limits_{j<n+1} s_j \subseteq A_n$) and $\langle t\rangle$ is disjoint from C_{n+1}. It is easy to see that this is a winning strategy. ⊣

Since every game with Borel payoff is determined, we can give a characterization of the Borel ideals that are Shelah-Steprāns.

Theorem 3.8. *If $\mathcal{I}$ is a Borel ideal then $\mathcal{I}$ is Shelah-Steprāns if and only if* $\mathsf{fin}\times\mathsf{fin} \leq_K \mathcal{I}$.

We can extend this theorem under some large cardinal assumptions. Fix a tree T of height ω, $f : [T] \longrightarrow \wp(\omega)$ a continuous function (where $[T]$ denotes the set of branches of T) and $\mathcal{W} \subseteq \wp(\omega)$. We then define the game $\mathcal{G}(T, f, \mathcal{W})$ as follows:

I	...	x_n		...
II	...		y_n	...

At round $n \in \omega$ player I plays x_n and II responds with y_n with the requirement that $\langle x_0, y_0, ..., x_n, y_n\rangle \in T$. Player I wins if $f(b) \in \mathcal{W}$ where b is the branch constructed during the game. The following is a well known extension of Martin's result (see [45]):

Proposition 3.9 (LC)**.** *If $\mathcal{W} \in \mathsf{L}(\mathbb{R})$ then $\mathcal{G}(T, f, \mathcal{W})$ is determined ($\mathsf{L}(\mathbb{R})$ denotes the smallest transitive model of* ZFC *that contains all reals)*

Here LC denotes a large cardinal assumption. In this case, it is enough to assume that there is a proper class of Woodin cardinals. The reader may consult the first chapter of [45]) for more information. We can conclude the following:

Theorem 3.10 (LC). *(1) Let $\mathcal{I} \in \mathsf{L}(\mathbb{R})$ be an ideal on ω. Then $\mathcal{I}$ is Shelah-Steprāns if and only if $\mathsf{fin}\times\mathsf{fin} \leq_K \mathcal{I}$.*

(2) Let $\mathcal{J}$ be a σ-ideal in ω^ω such that $\mathcal{J} \in \mathsf{L}(\mathbb{R})$ and $X \in tr(\mathcal{J})^+$. Then $tr(\mathcal{J}) \restriction X$ is Shelah-Steprāns if and only if $\mathsf{fin}\times\mathsf{fin} \leq_K tr(\mathcal{J}) \restriction X$.

Proof. To prove the first item, let Y be the set of all sequences $\langle A_0, s_0, ..., A_n, s_n \rangle$ such that $A_n \in \mathcal{I}$ and $s_n \in [\omega \setminus A_n]^{<\omega}$ and $\max(s_i) \subseteq A_{i+1}$ if $i < n$. Let T be the tree obtained by closing Y under restrictions. We define $f : [T] \longrightarrow \wp(\omega)$ by $f(b) = \bigcup_{n\in\omega} b(2n+1)$ where $b \in [T]$. Clearly $\mathcal{L}(\mathcal{I})$ is a game equivalent to $\mathcal{G}(T, f, \mathcal{I})$, so the result follows from the previous results. The second item is a consequence of the first. ⊣

The following result will be useful later, see Lemma 4.5:

Lemma 3.11. *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent:*

(1) $\mathcal{I}$ is Shelah-Steprāns.

(2) For every $\{X_n \mid n \in \omega\} \subseteq (\mathcal{I}^{<\omega})^+$ there is $B \in \mathcal{I}$ such that $X_n \cap [B]^{<\omega}$ is infinite for every $n \in \omega$.

Proof. Clearly 2 implies 1 and if 2 fails then it is easy to see that Player II has a winning strategy in $\mathcal{L}(\mathcal{I})$, so 1 also fails. ⊣

4. Strong properties of MAD families

In this section, we will study some strong combinatorial properties of MAD families and we will clarify the relationship between them. The basic notions and implications will be presented in the more general context of ideals (Subsection 4.1), and for existence, non-existence (see Subsections 4.3, 4.4 and 4.5), and non-implications (Subsection 4.6) we will consider the special case of MAD families. We shall also consider generic MAD families (Subsection 4.2).

4.1. Combinatorial properties of ideals: definitions and implications.

We start with:

Definition 4.1. *Let $\mathcal{I}$ be an ideal on ω.*

(1) $\mathcal{I}$ is tight *if for every $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}^+$ there is $A \in \mathcal{I}$ such that $A \cap X_n \neq \emptyset$ for every $n \in \omega$.*

(2) $\mathcal{I}$ is weakly tight *if for every $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}^+$ there is $A \in \mathcal{I}$ such that $|A \cap X_n| = \omega$ for infinitely many $n \in \omega$.*

(3) $\mathcal{I}$ is strongly tight *if for every $\{X_n \mid n \in \omega\} \subseteq [\omega]^\omega$ such that $\{n \mid X_n \subseteq^* Y\}$ is finite for every $Y \in \mathcal{I}$, there is $A \in \mathcal{I}$ such that $A \cap X_n \neq \emptyset$ for every $n \in \omega$.*

(4) We say a family $X = \{X_n \mid n \in \omega\}$ such that $X_n \subseteq [\omega]^{<\omega}$ is locally finite according to $\mathcal{I}$ *if for every $B \in \mathcal{I}$ for almost all $n \in \omega$ there is $s \in X_n$ such that $s \cap B = \emptyset$.*

(5) $\mathcal{I}$ is raving *if for every family $X = \{X_n \mid n \in \omega\}$ that is locally finite according to $\mathcal{I}$ there is $A \in \mathcal{I}$ such that A contains at least one element of each X_n.*

Obviously, strongly tight implies tight, which in turn implies weakly tight. Also, it is easy to see that raving implies both Shelah-Steprāns and strongly tight. Furthermore, by Lemma 3.11, every Shelah-Steprāns ideal is tight. We shall see later on (in particular in Subsection 4.6) that all these properties are (consistently) distinct, even for MAD families.

For a MAD family $\mathcal{A}$ and a property X of ideals, we say $\mathcal{A}$ *has property X* whenever the corresponding ideal $\mathcal{I}(\mathcal{A})$ has property X.

Tight MAD families are Cohen indestructible. Although Cohen indestructibility does not imply tightness, it is true that every Cohen indestructible MAD family has a restriction that is tight (see [20]). Thus, existence-wise, the two properties are on the same level.

In [30] and [39] it was proved that weakly tight MAD families exist under $\mathfrak{s} \leq \mathfrak{b}$. Strongly tight MAD families have the following characterization.

Lemma 4.2. *$\mathcal{A}$ is strongly tight iff whenever $\{B_n \mid n \in \omega\} \subseteq [\omega]^\omega$ is such that there is $\{A_n \mid n \in \omega\} \subseteq \mathcal{A}$ such that*

- *$B_n \subseteq A_n$ for all n,*
- *for all $A \in \mathcal{A}$, the set $\{n \mid A_n = A\}$ is finite,*

there is $A \in \mathcal{I}(\mathcal{A})$ such that $A \cap B_n \neq \emptyset$ for all n.

Proof. First assume $\mathcal{A}$ is strongly tight. Assume B_n is given as in the lemma. Clearly $X_n = B_n$ satisfies the assumption in the definition of "strongly tight" and we obtain $A \in \mathcal{I}(\mathcal{A})$ as required.

On the other hand, if X_n are given as in the definition of "strongly tight", first use the maximality of $\mathcal{A}$ to find $A_n \in \mathcal{A}$ such that $A_n \cap X_n$ is infinite and let $B_n = A_n \cap X_n$. Whenever possible, choose A_n distinct from $A_i, i < n$. The assumption on the X_n then guarantees that every $A \in \mathcal{A}$ is chosen only finitely often. The $A \in \mathcal{I}(\mathcal{A})$ given by the lemma is as required. $\dashv$

Like the Shelah-Steprāns property, the properties considered here are upwards closed in the Katětov order (see Lemma 3.4).

Lemma 4.3. *Assume $\mathcal{I}$ and $\mathcal{J}$ are ideals on ω.*

1. *If $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{I}$ is tight (weakly tight, raving, resp.), then so is $\mathcal{J}$.*
2. *If $\mathcal{I} \leq_{KB} \mathcal{J}$ and $\mathcal{I}$ is strongly tight, then so is $\mathcal{J}$.*
3. *Assume $\mathcal{A}$ and $\mathcal{B}$ are MAD families. If $\mathcal{A}$ is strongly tight and $\mathcal{I}(\mathcal{A}) \leq_K \mathcal{I}(\mathcal{B})$ then $\mathcal{B}$ is strongly tight.*

Proof. 1. This is a standard argument. Let $f : \omega \to \omega$ be a Katětov reduction. Assume first $\mathcal{I}$ is tight. Take $\{X_n \mid n \in \omega\} \subseteq \mathcal{J}^+$ and let $Y_n = f[X_n]$. Clearly the Y_n belong to $\mathcal{I}^+$, and therefore there is $B \in \mathcal{I}$ such that $B \cap Y_n \neq \emptyset$ for all n. Letting $A = f^{-1}[B] \in \mathcal{J}$ we see that $A \cap X_n \neq \emptyset$ for all n. The proofs for "weakly tight" and "raving" are similar.

2. This is also similar.

3. Fix a Katětov-morphism f from $(\omega, \mathcal{I}(\mathcal{B}))$ to $(\omega, \mathcal{I}(\mathcal{A}))$ and a family $\mathcal{W} = \{X_n \mid n \in \omega\}$ such that for every $n \in \omega$ there is $B_n \in \mathcal{B}$ such that $X_n \subseteq B_n$ and for every $B \in \mathcal{B}$ the set $\{n \mid B_n = B\}$ is finite. Let $\mathcal{W}_1 = \left\{X \in \mathcal{W} \mid f[X] \in [\omega]^{<\omega}\right\}$ and for every $X \in \mathcal{W}_1$ we choose $b_X \in f[X]$ such that $f^{-1}(\{b_X\})$ is infinite. We first claim that the set $Y = \{b_X \mid X \in \mathcal{W}_1\}$ is finite. If this was not the case, we could find $A \in \mathcal{A}$ such that $A \cap Y$ is infinite. Since f is a Katětov-morphism, we conclude that $f^{-1}(A) \in \mathcal{I}(\mathcal{B})$ and $\left\{X \in \mathcal{W} \mid f^{-1}(A) \cap X \in [\omega]^\omega\right\}$ is infinite, but this is a contradiction. Using that Y is finite, it is easy to see that $\mathcal{W}_1$ must also be finite.

Letting $\mathcal{W}_2 = \mathcal{W} \setminus \mathcal{W}_1$, for every $X \in \mathcal{W}_2$ we choose $A_X \in \mathcal{I}(\mathcal{A})$ such that $Y_X = A_X \cap f[X]$ is infinite. Note that if $A \in \mathcal{A}$ then the set $\{X \in \mathcal{W}_2 \mid A = A_X\}$ must be finite. Since $\mathcal{A}$ is strongly tight we can find $A \in \mathcal{I}(A)$ such that $A \cap Y_X \neq \emptyset$ for every $X \in \mathcal{W}_2$. Since f is a Katětov-morphism, we may conclude that $B = f^{-1}(A)$ belongs to $\mathcal{I}(\mathcal{B})$ and $B \cap X \neq \emptyset$ for every $X \in \mathcal{W}_2$. Clearly $B \cup \bigcup \mathcal{W}_1$ has the desired properties. $\dashv$

⋆ ⋆ ⋆

We next define properties of ideals $\mathcal{I}$ relevant for the investigation of Mathias forcing $\mathbb{M}(\mathcal{I})$:

Definition 4.4. *Let $\mathcal{I}$ be an ideal in ω.*

(1) We say $\mathcal{I}$ is Canjar *if and only if for every $\{X_n \mid n \in \omega\} \subseteq (\mathcal{I}^{<\omega})^+$ there are $Y_n \in [X_n]^{<\omega}$ such that $\bigcup_{n\in\omega} Y_n \in (\mathcal{I}^{<\omega})^+$.*

(2) We say $\mathcal{I}$ is Hurewicz *if and only if for every $\{X_n \mid n \in \omega\} \subseteq (\mathcal{I}^{<\omega})^+$ there are $Y_n \in [X_n]^{<\omega}$ such that $\bigcup_{n\in A} Y_n \in (\mathcal{I}^{<\omega})^+$ for every $A \in [\omega]^\omega$.*

Clearly, every Hurewicz ideal is Canjar. Moreover, every F_σ ideal is Hurewicz [4], and every Borel Canjar ideal is F_σ [10, 15], so that for Borel ideals, F_σ, Canjar, and Hurewicz agree. In general, the two notions are quite different: Canjar [9] constructed *Canjar ultrafilters* (i.e. ultrafilters whose dual maximal ideals are Canjar) under CH, while it is easy to see that no maximal ideal can be Hurewicz.

If $\mathcal{I}$ is an ideal, we denote by $\mathbb{M}(\mathcal{I})$ the *Mathias forcing with* $\mathcal{I}$, that is, the set of all pairs (s, A) such that $s \in [\omega]^{<\omega}$ and $A \in \mathcal{I}$, ordered by $(s, A) \leq (t, B)$ if $t \subseteq s$, $B \subseteq A$ and $(s \setminus t) \cap B = \emptyset$, where $(s, A), (t, B) \in \mathbb{M}(\mathcal{I})$. It is easy to see that $\mathbb{M}(\mathcal{I})$ destroys the tallness of $\mathcal{I}$.

We mention the following important results regarding Canjar and Hurewicz ideals:

- $\mathcal{I}$ is Canjar if and only if $\mathbb{M}(\mathcal{I})$ does not add a dominating real [21].
- $\mathcal{I}$ is Hurewicz if and only if $\mathbb{M}(\mathcal{I})$ preserves all unbounded families of the ground model [10].
- $\mathcal{I}$ is Canjar if and only if $\mathcal{I}$ is a Menger subspace of $\wp(\omega)$ [10].
- $\mathcal{I}$ is Hurewicz if and only if $\mathcal{I}$ is a Hurewicz subspace of $\wp(\omega)$ [10].

For MAD families we have:

Proposition 4.5. *Every Shelah-Steprāns MAD family is Hurewicz.*

Proof. Let $\mathcal{A}$ be a Shelah-Steprāns MAD family and $\{X_n \mid n \in \omega\} \subseteq \left(\mathcal{I}(\mathcal{A})^{<\omega}\right)^+$. Note that if $B \in \mathcal{I}(\mathcal{A})$ then $\left\{X_n \setminus [B]^{<\omega} \mid n \in \omega\right\} \subseteq \left(\mathcal{I}(\mathcal{A})^{<\omega}\right)^+$. Using the Lemma 3.11, we can thus recursively find $\{B_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})$ with the following properties:

(1) If $n \neq m$ then there is no $A \in \mathcal{A}$ that has infinite intersection with both B_n and B_m.

(2) $X_n \cap [B_m \setminus k]^{<\omega}$ is infinite for all $n, m, k \in \omega$.

For every $n \in \omega$ let $Y_n \in [X_n]^{<\omega}$ such that $Y_n \cap [B_i]^{<\omega} \neq \emptyset$ for every $i \leq n$. It is then easy to see that if $D \in [\omega]^\omega$ then $\bigcup_{n\in D} Y_n \in \left(\mathcal{I}(\mathcal{A})^{<\omega}\right)^+$. ⊣

This implication clearly fails for ideals in general because, for example, $\mathsf{fin}\times\mathsf{fin}$ is Shelah-Steprāns, but not Canjar.

⋆⋆⋆

We will consider some more properties of ideals:

Definition 4.6. *Let $\mathcal{J}$ be an ideal.*

(1) $\mathcal{J}$ is Laflamme *if $\mathcal{J}$ can not be extended to an F_σ ideal.*

(2) $\mathcal{J}$ is not-P *if $\mathcal{J}$ can not be extended to an analytic P-ideal.*

(3) $\mathcal{J}$ is $\mathsf{fin}\times\mathsf{fin}$-like *if $\mathcal{J} \not\leq_K \mathcal{I}$ for every analytic ideal $\mathcal{I}$ such that $\mathsf{fin}\times\mathsf{fin} \not\leq_K \mathcal{I}$.*

By results of Solecki, Laczkovich and Recław it can be proved that no $F_{\sigma\delta}$ ideal is Katětov above $\mathsf{fin}\times\mathsf{fin}$ (see [44] and [25]). Since every analytic P-ideal is $F_{\sigma\delta}$ (see

[43]), it follows that every fin×fin-like ideal is not-P. Furthermore, in [32] it was proved that every F_σ ideal is contained in an analytic P-ideal, so every not-P ideal is Laflamme. By Theorem 3.10, every Shelah-Steprāns ideal is fin×fin-like.[3]

We have the following:

Lemma 4.7. *Let $\mathcal{J}$ be an ideal.*

1. *$\mathcal{J}$ is Laflamme if and only if $\mathcal{J} \nleq_K \mathcal{I}$ for every F_σ-ideal $\mathcal{I}$.*
2. *$\mathcal{J}$ is not-P if and only if $\mathcal{J} \nleq_{KB} \mathcal{I}$ for every analytic P-ideal $\mathcal{I}$.*

Proof. Clearly if $\mathcal{J} \nleq_K \mathcal{I}$ for every F_σ-ideal $\mathcal{I}$ then $\mathcal{J}$ is Laflamme. So assume that $\mathcal{J}$ is Laflamme, let $\mathcal{I}$ be an F_σ ideal, and let $f : \omega \longrightarrow \omega$. We must show that f is not a Katětov-morphism from $(\omega, \mathcal{I})$ to $(\omega, \mathcal{J})$. Define $\mathcal{I}' = \left\{X \mid f^{-1}(X) \in \mathcal{I}\right\}$. Let φ be a lower semicontinuous submeasure such that $\mathcal{I} = \mathsf{fin}(\varphi)$. For every $n \in \omega$ we define $C_n = \left\{X \mid \varphi\left(f^{-1}(X)\right) \leq n\right\}$. It is easy to see that each C_n is a closed set and $\mathcal{I}' = \bigcup\limits_{n\in\omega} C_n$. Since $\mathcal{J}$ is not contained in $\mathcal{I}'$ the result follows.

For the second part, it is clear that if $\mathcal{J} \nleq_{KB} \mathcal{I}$ for every analytic P-ideal $\mathcal{I}$ then $\mathcal{J}$ is not-P. Assume $\mathcal{J}$ is not-P, let $\mathcal{I}$ be an analytic P-ideal, and let $f : \omega \longrightarrow \omega$ be finite to one. We must show that f is not a Katětov-morphism from $(\omega, \mathcal{I})$ to $(\omega, \mathcal{J})$. Let φ be a lower semicontinuous submeasure such that $\mathcal{I} = \mathsf{Exh}(\varphi)$. Define $\sigma : \wp(\omega) \longrightarrow \mathbb{R} \cup \{\infty\}$ by $\sigma(A) = \varphi\left(f^{-1}(A)\right)$. It is easy to see that σ is a lower semicontinuous submeasure (it is a submeasure by definition and the lower semicontinuity follows since f is finite to one). Since $\mathcal{J}$ is not-P, there is $B \in \mathcal{J}$ such that $B \notin \mathsf{Exh}(\sigma)$ which implies that $f^{-1}(B) \notin \mathcal{I}$. ⊣

Note in this context that we could have defined "fin×fin-like" as we did define "Laflamme" and "not-P": namely, $\mathcal{J}$ is fin×fin-like if $\mathcal{J} \nsubseteq \mathcal{I}$ for every analytic ideal $\mathcal{I}$ such that $\mathsf{fin}\times\mathsf{fin} \nleq_K \mathcal{I}$. To see the nontrivial direction of this equivalence, assume there is an analytic ideal $\mathcal{I}$ such that $\mathsf{fin}\times\mathsf{fin} \nleq_K \mathcal{I}$ and $\mathcal{J} \leq_K \mathcal{I}$ as witnessed by the Katětov reduction f. Define $\mathcal{I}' = \{A | f^{-1}[A] \in \mathcal{I}\}$ and note that $\mathcal{J} \subseteq \mathcal{I}' \leq_K \mathcal{I}$, that $\mathsf{fin}\times\mathsf{fin} \nleq_K \mathcal{I}'$ and that $\mathcal{I}'$ is still analytic.

Also notice that all three properties defined here are (trivially) upwards closed in the Katětov order.

We now investigate the connection between fin×fin-likeness and weak tightness. Define $C_n = \{(n, m) \mid m \in \omega\}$.

Definition 4.8. *We define the ideal $\mathcal{WT}$ on $\omega \times \omega$ as follows:*

1. *$\mathcal{WT} \upharpoonright C_n$ is a copy of fin×fin (for every $n \in \omega$).*
2. *$A \in \mathcal{WT}$ iff $A \cap C_n \in \mathcal{WT} \upharpoonright C_n$ for all n and $A \cap C_n$ is finite for all but finitely many n.*

Clearly, $\mathcal{WT}$ is not weakly tight as witnessed by the C_n. Also note that if $B \subseteq \omega \times \omega$ has infinite intersection with infinitely many columns then $B \in \mathcal{WT}^+$.

Proposition 4.9. *$\mathcal{WT}$ is strictly Katětov below fin×fin.*

Proof. Note that the identity mapping witnesses $\mathcal{WT} \leq_K \mathsf{fin}\times\mathsf{fin}$. Now, we will show II has a winning strategy in $\mathcal{L}(\mathcal{WT})$. This is easy, since no C_n belongs to $\mathcal{WT}$ and therefore II can play in such a way that the set she constructed at the end intersects infinitely often all the C_n, so it can not be an element of $\mathcal{WT}$. ⊣

For MAD families we have the following characterization:

Lemma 4.10. *If $\mathcal{A}$ is a MAD family then $\mathcal{A}$ is weakly tight if and only if $\mathcal{I}(\mathcal{A}) \nleq_K \mathcal{WT}$.*

[3] This uses LC; note that if in the definition of fin×fin-like we only quantified over Borel ideals, then this implication would be true in ZFC by Theorem 3.8.

Proof. We will prove that $\mathcal{A}$ is not weakly tight if and only if $\mathcal{I}(\mathcal{A}) \leq_K \mathcal{WT}$. One direction follows from the fact that $\mathcal{WT}$ is not weakly tight and from the upwards closure of weak tightness in the Katětov order (see Lemma 4.3).

So assume $\mathcal{A}$ is not weakly tight. So there is a partition $X = \{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ such that if $A \in \mathcal{A}$ then $A \cap X_n$ is finite for almost all $n \in \omega$. Since $\mathcal{A} \restriction X_n$ is an AD family, we know $\mathcal{A} \restriction X_n \leq_{KB} \mathsf{fin}\times\mathsf{fin}$, so for each $n \in \omega$ fix a Katětov-Blass-morphism $h_n : C_n \longrightarrow X_n$ from $(C_n, \mathcal{WT} \restriction C_n)$ to $(X_n, \mathcal{A} \restriction X_n)$ (in fact, it is well-known that we can choose one to one h_n). Letting $h = \bigcup h_n$ we will show h is a Katětov-morphism from $(\omega \times \omega, \mathcal{WT})$ to $(\omega, \mathcal{A})$. If $A \in \mathcal{A}$ then we can find $B \in X^{\perp}$ and a finite set $F \subseteq \omega$ such that $A = \bigcup_{n\in F} (A \cap X_n) \cup B$. Clearly $h^{-1}(B) \in \mathcal{WT}$ since $h^{-1}(B) \in \emptyset\times\mathsf{fin}$ and $h^{-1}(A \cap X_n) = h_n^{-1}(A \cap X_n)$ which is an element of $\mathcal{WT}$ since h_n is a Katětov-morphism. Therefore $h^{-1}(A) \in \mathcal{WT}$. ⊣

We conclude:

Corollary 4.11. *If $\mathcal{A}$ is $\mathsf{fin}\times\mathsf{fin}$-like then $\mathcal{A}$ is weakly tight.*

This implication fails for ideals in general:

Proposition 4.12. *There is a $\mathsf{fin}\times\mathsf{fin}$-like ideal that is not weakly tight.*

Proof. Let $\mathcal{J}$ be any ideal not contained in an analytic ideal (e.g., any maximal ideal). Define the ideal $\mathcal{WT}(\mathcal{J})$ on $\omega \times \omega$ as follows:

1. $\mathcal{WT}(\mathcal{J}) \restriction C_n$ is a copy of $\mathcal{J}$.
2. $A \in \mathcal{WT}(\mathcal{J})$ iff $A \cap C_n \in \mathcal{WT}(\mathcal{J}) \restriction C_n$ for all n and $A \cap C_n$ is finite for all but finitely many n.

As for $\mathcal{WT}$ we see that the C_n witness that $\mathcal{WT}(\mathcal{J})$ is not weakly tight. Also note that as in Proposition 4.9 we see that $\mathcal{WT}(\mathcal{J})$ is strictly Katětov below $\mathsf{fin}\times\mathsf{fin}$. To see that $\mathcal{WT}(\mathcal{J})$ is $\mathsf{fin}\times\mathsf{fin}$-like, let $\mathcal{I}$ be an analytic ideal on $\omega \times \omega$ with $\mathsf{fin}\times\mathsf{fin} \not\leq_K \mathcal{I}$. We need to see that $\mathcal{WT}(\mathcal{J}) \not\subseteq \mathcal{I}$. Since $\mathsf{fin}\times\mathsf{fin} \not\leq_K \mathcal{I}$ there is $A \in \mathsf{fin}\times\mathsf{fin}$ with $A \notin \mathcal{I}$, say $A = B \cup C$ where B meets only finitely many C_n and $C \cap C_n$ is finite for all n. If $C \notin \mathcal{I}$, we are done because $C \in \mathcal{WT}(\mathcal{J})$. So assume $B \notin \mathcal{I}$. Thus, for some n, $B \cap C_n \notin \mathcal{I}$ and $\mathcal{I} \restriction C_n$ is a proper analytic ideal. Then $\mathcal{WT}(\mathcal{J}) \restriction C_n \not\subseteq \mathcal{I} \restriction C_n$, and we are also done. ⊣

⋆ ⋆ ⋆

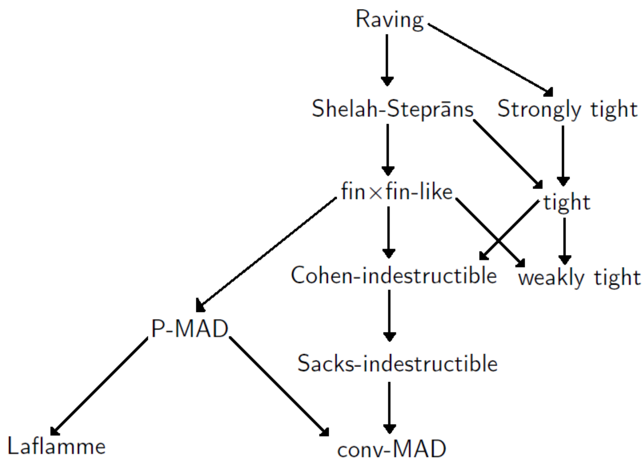
We briefly discuss the connection between the properties introduced so far and indestructibility by forcing:

Proposition 4.13. *Let $\mathcal{I}$ be an ideal. Also let $\mathcal{J}$ be a σ-ideal in ω^ω for which the forcing $\mathbb{P}_{\mathcal{J}} = Borel(\omega^\omega)/\mathcal{J}$ is proper, has the continuous reading of names and does not add a dominating real (below any condition).*

1. *Assume $\mathcal{I}$ is $\mathsf{fin}\times\mathsf{fin}$-like and $tr(\mathcal{J})$ is an analytic ideal. Then $\mathcal{I}$ is $\mathbb{P}_{\mathcal{J}}$-indestructible.*
2. *(LC) If $\mathcal{I}$ is Shelah-Steprāns and $\mathcal{J} \in \mathsf{L}(\mathbb{R})$ then $\mathcal{I}$ is $\mathbb{P}_{\mathcal{J}}$-indestructible.*
3. *In particular, if $\mathcal{I}$ is Shelah-Steprāns (or just $\mathsf{fin}\times\mathsf{fin}$-like), then $\mathcal{I}$ is Cohen, random, and Sacks indestructible.*
4. *If $\mathcal{I}$ is a not-P ideal then $\mathcal{I}$ is random and Sacks indestructible.*

Proof. 1. Let $\mathcal{J}$ be a σ-ideal in ω^ω such that $\mathbb{P}_{\mathcal{J}}$ is proper and has the continuous reading of names. If there is $B \in \mathbb{P}_{\mathcal{J}}$ such that forcing below B destroys $\mathcal{I}$, then there is $X \in tr(\mathcal{J})^+$ such that $\mathcal{I} \leq_K tr(\mathcal{J}) \restriction X$ (see Proposition 1.4). Since $\mathcal{I}$ is $\mathsf{fin}\times\mathsf{fin}$-like and $tr(\mathcal{J})$ is analytic, $\mathsf{fin}\times\mathsf{fin} \leq_{\mathsf{K}} tr(\mathcal{J}) \restriction X$ follows, and so $\mathbb{P}_{\mathcal{J}}$ must add a dominating real below some condition.

2. This is similar, using Theorem 3.10.



3. Noting that the ideals $tr(\mathcal{M})$, $tr(\mathcal{N})$, and $tr(\mathsf{ctble})$ are all Borel, this follows from the previous items.

4. In [16, Theorem 3.4] it was proved that $tr(\mathcal{N}) \leq_K \mathcal{Z}$. So every not-P ideal is random indestructible. It is well known that random indestructibility entails Sacks indestructibility (because $tr(\mathsf{ctble}) \leq_K tr(\mathcal{N})$, see also [8]). $\dashv$

4.2. **Generic MAD families.** Let $\mathcal{I}$ be a (perhaps improper) tall ideal. We define $\mathbb{P}_{\mathsf{MAD}}(\mathcal{I})$ as the set of all countable AD families contained in $\mathcal{I}$, ordered by inclusion. It is easy to see that this is a σ-closed forcing adding a MAD family contained in $\mathcal{I}$, which we will denote by $\mathcal{A}_{gen}(\mathcal{I})$. By $\mathbb{P}_{\mathsf{MAD}}$ we denote $\mathbb{P}_{\mathsf{MAD}}(\wp(\omega))$, which is the set of all countable AD families ordered by inclusion; we will denote by $\mathcal{A}_{gen}$ the generic MAD family. We shall use such generic objects later for non-implications (see Subsection 4.6).

Definition 4.14. *Let $\mathcal{I}$ be an ideal. We say that $\mathcal{I}$ is* nowhere Shelah-Steprāns *if no restriction of $\mathcal{I}$ is Shelah-Steprāns.*

It is easy to see that nwd, $tr(\mathsf{ctble})$, $tr(\mathcal{N})$, $tr(\mathcal{K}_\sigma)$ and every F_σ-ideal are nowhere Shelah-Steprāns.

Lemma 4.15. *Let $\mathcal{I}, \mathcal{J}$ be two ideals such that $\mathcal{I}$ is nowhere Shelah-Steprāns and $\mathcal{J} \nleq_K \mathcal{I}$. Let $\mathcal{A} \subseteq \mathcal{J}$ be a countable AD family and let $f : (\omega, \mathcal{I}) \longrightarrow (\omega, \mathcal{I}(\mathcal{A}))$ be a Katětov-morphism. Then there is $B \in \mathcal{J} \cap \mathcal{A}^{\perp}$ such that $f^{-1}(B) \in \mathcal{I}^{+}$.*

Proof. Let $\mathcal{A} = \{A_n \mid n \in \omega\}$. We know f is a Katětov-morphism, so the set $\{f^{-1}(A_n) \mid n \in \omega\}$ is contained in $\mathcal{I}$. Since $\mathcal{J} \nleq_K \mathcal{I}$ there is $D \in \mathcal{J}$ such that $C = f^{-1}(D) \in \mathcal{I}^{+}$. Since $\mathcal{I} \restriction C$ is not Shelah-Steprāns, there is $X \in \left((\mathcal{I} \restriction C)^{<\omega}\right)^{+}$ such that no element of $\mathcal{I}$ contains infinitely many elements of X. For each $n \in \omega$ we choose $s_n \in X$ such that $s_n \cap \left(f^{-1}(A_0 \cup ... \cup A_n)\right) = \emptyset$. We then know that $E = \bigcup s_n \in \mathcal{I}^{+}$. It is easy to see that $B = f[E]$ has the desired properties. $\dashv$

We conclude:

Corollary 4.16. *Let $\mathcal{I}, \mathcal{J}$ be ideals such that $\mathcal{I}$ is nowhere Shelah-Steprāns and $\mathcal{J} \nleq_K \mathcal{I}$.*

(1) *$\mathbb{P}_{\mathsf{MAD}}(\mathcal{J})$ forces that $\mathcal{A}_{gen}(\mathcal{J})$ is not Katětov below $\mathcal{I}$.*
(2) *The Continuum Hypothesis implies that there is a MAD family $\mathcal{A} \subseteq \mathcal{J}$ such that $\mathcal{I}(\mathcal{A}) \nleq_K \mathcal{I}$.*

In particular, $\mathcal{A}_{gen}(\mathsf{nwd})$ is a Cohen destructible MAD family that is Miller and random indestructible. For more on this type of results, the reader may consult [8].

Proposition 4.17. $\mathbb{P}_{\mathsf{MAD}}$ *forces that* $\mathcal{A}_{gen}$ *is raving.*

Proof. Let $\mathcal{B} \in \mathbb{P}_{\mathsf{MAD}}$ and $X = \{X_n \mid n \in \omega\}$ such that $\mathcal{B}$ forces that X is locally finite according to $\mathcal{I}(\mathcal{A}_{gen})$. Let $\mathcal{B} = \{B_n \mid n \in \omega\}$ and we define $E_n = B_0 \cup ... \cup B_n$ for every $n \in \omega$. We can then find an interval partition $\mathcal{P} = \{P_n \mid n \in \omega\}$ of ω such that if $i \in P_{n+1}$ then E_n does not intersect every element of X_i. For every $i \in \omega$ we choose $s_i \in X_i$ as follows: if $i \in P_0$ let s_i be any element of X_i and if $i \in P_{n+1}$ we choose $s_i \in X_i$ such that $s_i \cap E_n = \emptyset$. Let $A = \bigcup_{n \in \omega} s_n$; then $A \in \mathcal{B}^{\perp}$ and the condition $\mathcal{B} \cup \{A\} \in \mathbb{P}_{\mathsf{MAD}}$ is the extension of $\mathcal{B}$ we were looking for. ⊣

Our motivation for studying the forcing $\mathbb{P}_{\mathsf{MAD}}$ comes from the following results about generic ultrafilters:

Theorem 4.18 (Todorcevic, see [12])**.** *An ultrafilter* $\mathcal{U}$ *is* $\wp(\omega) \setminus \mathsf{fin}$ *generic over* $\mathsf{L}(\mathbb{R})$ *if and only if* $\mathcal{U}$ *is Ramsey.*

Theorem 4.19 (Chodounský, Zapletal, see [11])**.** *Let* $\mathcal{I}$ *be an* F_σ*-ideal and* $\mathcal{U}$ *an ultrafilter.* $\mathcal{U}$ *is* $\wp(\omega) \setminus \mathcal{I}$ *generic over* $\mathsf{L}(\mathbb{R})$ *if and only if* $\mathcal{I} \cap \mathcal{U} = \emptyset$ *and for every closed set* $\mathcal{C}$ *if* $\mathcal{C} \cap \mathcal{U} = \emptyset$ *then there is* $A \in \mathcal{U}$ *such that* $A \cap Y \in \mathcal{I}$ *for every* $Y \in \mathcal{C}$*.*

It would be interesting to find a similar characterization of the $\mathbb{P}_{\mathsf{MAD}}$ generics over $\mathsf{L}(\mathbb{R})$:

Problem 4.20. *Is there a combinatorial characterization of* $\mathcal{A}$ *(or of* $\mathcal{I}(\mathcal{A})$*) where* $\mathcal{A}$ *is* $\mathbb{P}_{\mathsf{MAD}}$ *generic over* $\mathsf{L}(\mathbb{R})$*?*

A natural candidate would be "raving", but even this strong property might still be too weak to capture the full extent of genericity.

4.3. **Existence versus non-existence.** An important result of Raghavan says:

Theorem 4.21 ([36])**.** *It is consistent that there are no Shelah-Steprāns* MAD *families.*

For strongly tight families, we first prove:

Proposition 4.22. *If* $\mathcal{A}$ *is strongly tight then* $\mathfrak{d} \leq |\mathcal{A}|$.

Proof. Let $\{A_n \mid n \in \omega\}$ be a partition of ω contained in $\mathcal{A}$ and for each $n \in \omega$ let $P_n = \{A_n(i) \mid i \in \omega\}$ be a partition of A_n in infinite pieces. Given $A \in \mathcal{I}(\mathcal{A})$ we define a function $f_A : \omega \longrightarrow \omega$ given by $f_A(n) = 0$ if $A \cap A_n$ is infinite and in the other case $f_A(n) = \max\{i \mid A \cap A_n(i) \neq \emptyset\} + 1$. We claim that $\{f_A \mid A \in \mathcal{I}(\mathcal{A})\}$ is a dominating family. Assume this is not the case, so there is $g : \omega \longrightarrow \omega$ not dominated by any of the f_A. For each $n \in \omega$ define $X_n = A_n(g(n))$ and $X = \{X_n \mid n \in \omega\}$. Since $\mathcal{A}$ is strongly tight there must be $A \in \mathcal{I}(\mathcal{A})$ such that $A \cap X_n \neq \emptyset$ for every $n \in \omega$. Pick any m such that $f_A(m) < g(m)$; then $A \cap A_m(g(m)) = \emptyset$ so that $A \cap X_m = \emptyset$, which is a contradiction. ⊣

We conclude:

Corollary 4.23. *There are no strongly tight* MAD *families in the Cohen model.*

Proof. If there were a strongly tight MAD, it would have size continuum, by the previous proposition. But since it would also be tight, it should have size ω_1 (recall that tight MAD families are Cohen indestructible). ⊣

We will later prove that there are Shelah-Steprāns MAD families in the Cohen model, so Shelah-Steprāns does not imply strong tightness (see the discussion after Theorem 4.27). As mentioned in the Introduction (Problem 1.1), it is still open whether tight MAD families exist in ZFC.

To provide an example for a class of MAD families existing in ZFC, we make the following definition:

Definition 4.24. *Let $\mathcal{I}$ be an ideal and $\mathcal{A}$ a MAD family. We say that $\mathcal{A}$ is $\mathcal{I}$-MAD if $\mathcal{I}(\mathcal{A}) \nleq_K \mathcal{I}$.*

It is well known that no MAD family is $\mathsf{fin}\times\mathsf{fin}$-MAD. Clearly $\mathcal{A}$ is Laflamme iff it is $\mathcal{I}$-MAD for every F_σ-ideal $\mathcal{I}$, and if $\mathcal{A}$ is $\mathcal{I}$-MAD for every analytic P-ideal then $\mathcal{A}$ is not-P (Lemma 4.7). Also note that being Cohen indestructible is equivalent to being nwd-MAD (recall that nwd denotes the ideal of nowhere dense sets of the rational numbers).

We denote by conv the ideal in $[0,1]\cap\mathbb{Q}$ generated by all sequences converging to a real number. Since $\mathsf{conv} \leq_K tr(\mathsf{ctble})$, every Sacks indestructible MAD family is conv-MAD. On the other hand, if $\mathcal{A}$ is a $\mathbb{P}_{\mathsf{MAD}}(tr(\mathsf{ctble}))$-generic MAD family, then by Corollary 4.16, $\mathcal{A}$ is a Sacks destructible conv-MAD family because $tr(\mathsf{ctble}) \nleq_K \mathsf{conv}$. We will now prove that there is a conv-MAD family; this result is based on the proof of Proposition 2 of [17]. We need the following lemma:

Lemma 4.25. *Let $\mathcal{A}$ be an AD family of size less than $\mathfrak{c}$. If f is a Katětov-morphism from $([0,1]\cap\mathbb{Q},\mathsf{conv})$ to $(\omega,\mathcal{I}(\mathcal{A}))$ then there is $B\in\mathcal{A}^{\perp}$ such that $f^{-1}(B)\notin \mathsf{conv}$.*

Proof. For every $A\in\mathcal{A}$ let F_A be the set of accumulation points of $f^{-1}(A)$, and note that each F_A is finite since $f^{-1}(A)$ can be covered by finitely many converging sequences. Since $[0,1]$ can be partitioned into $\mathfrak{c}$-many perfect pairwise disjoint sets, we can find a perfect set $C\subseteq[0,1]$ such that $C\cap F_A=\emptyset$ for every $A\in\mathcal{A}$. Let $D\subseteq[0,1]\cap\mathbb{Q}$ be such that C is the set of accumulation points of D and note that $D\cap f^{-1}(A)$ is finite for all $A\in\mathcal{A}$. It is easy to see that $B=f[D]$ has the desired properties. $\dashv$

We conclude:

Corollary 4.26. *There is a conv-MAD family.*

The ideal conv is one of the few Borel ideals $\mathcal{I}$ for which we can prove that there are $\mathcal{I}$-MAD families.

4.4. **Existence under diamond principles.** Parametrized diamonds are strong guessing principles which can be used to construct MAD families with strong combinatorial properties. We first recall the principles $\diamondsuit(\mathfrak{b})$ and $\diamondsuit(\mathfrak{d})$ from [33].

- **$\diamondsuit(\mathfrak{b})$):** For every $\langle F_\alpha : 2^\alpha \longrightarrow \omega^\omega\rangle_{\alpha<\omega_1}$ such that each F_α is Borel, there is $g:\omega_1\longrightarrow\omega^\omega$ such that for every $R\in 2^{\omega_1}$, the set $\{\alpha \mid F_\alpha(R\restriction\alpha) \;{}^*\!\ngeq g(\alpha)\}$ is stationary.
- **$\diamondsuit(\mathfrak{d})$):** For every $\langle F_\alpha : 2^\alpha \longrightarrow \omega^\omega\rangle_{\alpha<\omega_1}$ such that each F_α is Borel, there is $g:\omega_1\longrightarrow\omega^\omega$ such that for every $R\in 2^{\omega_1}$, the set $\{\alpha \mid F_\alpha(R\restriction\alpha)\leq^* g(\alpha)\}$ is stationary.

Clearly $\diamondsuit(\mathfrak{d})$ implies $\diamondsuit(\mathfrak{b})$. In [33] it was proved that $\diamondsuit(\mathfrak{b})$ implies that $\mathfrak{a}=\omega_1$ and in [14] it was shown that $\diamondsuit(\mathfrak{b})$ implies the existence of a tight MAD family. We will now improve this result:

Theorem 4.27. *(1) $\diamondsuit(\mathfrak{b})$ implies there is a Shelah-Steprāns MAD family.*
(2) $\diamondsuit(\mathfrak{d})$ implies there is a raving MAD family.

Proof. For every $\alpha < \omega_1$ fix an enumeration $\alpha = \{\alpha_n \mid n \in \omega\}$. We will first show that $\diamondsuit(\mathfrak{b})$ implies there is a Shelah-Steprāns MAD family. With a suitable coding, the coloring C will be defined for pairs $t = (\mathcal{A}_t, X_t)$ where $\mathcal{A}_t = \langle A_\xi \mid \xi < \alpha \rangle$ and $X_t \subseteq [\omega]^{<\omega}$. We define $C(t)$ to be the constant 0 function in case $\mathcal{A}_t$ is not an almost disjoint family or $X_t \notin \left(\mathcal{I}(\mathcal{A}_t)^{<\omega}\right)^+$. In the other case, define an increasing function $C(t) : \omega \longrightarrow \omega$ such that if $n \in \omega$ then there is $s \in X_t$ such that $s \subseteq C(t)(n)$ and $s \cap (A_{\alpha_0} \cup ... \cup A_{\alpha_n} \cup n) = \emptyset$.

Using $\diamondsuit(\mathfrak{b})$ let $G : \omega_1 \longrightarrow \omega^\omega$ be a guessing sequence for C. By changing G if necessary, we may assume that all the $G(\alpha)$ are increasing and if $\alpha < \beta$ then $G(\alpha) <^* G(\beta)$. We will now define our MAD family: start by taking $\{A_n \mid n \in \omega\}$ a partition of ω. Having defined A_ξ for all $\xi < \alpha$, we proceed to define $A_\alpha = \bigcup\limits_{n\in\omega} (G(\alpha)(n) \setminus A_{\alpha_0} \cup ... \cup A_{\alpha_n})$ in case this is an infinite set, otherwise just take any A_α that is almost disjoint from $\{A_\beta \mid \beta < \alpha\}$. We will see that $\mathcal{A}$ is a Shelah-Steprāns MAD family. Let $X \in \left(\mathcal{I}(\mathcal{A})^{<\omega}\right)^+$. Consider the branch $R = (\langle A_\xi \mid \xi < \omega_1 \rangle, X)$ and pick $\beta > \omega$ such that $C(R \restriction \beta) \ ^* \ngeq G(\beta)$. It is easy to see that A_β contains infinitely many elements of X.

Now we will prove that $\diamondsuit(\mathfrak{d})$ implies there is a raving MAD family. With a suitable coding, the coloring C will be defined for pairs $t = (\mathcal{A}_t, X_t)$ where $\mathcal{A}_t = \langle A_\xi \mid \xi < \alpha \rangle$ and $X_t = \{X^t_n \mid n \in \omega\} \subseteq [\omega]^{<\omega}$. We define $C(t)$ to be the constant 0 function in case $\mathcal{A}_t$ is not an almost disjoint family or X_t is not locally finite according to $\mathcal{I}(\mathcal{A}_t)$. We will describe what to do in the other case. For every $n \in \omega$ define $B_n = \bigcup\limits_{i<n} A_{\alpha_i}$ (hence $B_0 = \emptyset$) and let $d(n)$ be the smallest $k \geq n$ such that if $\ell \geq k$ then B_n does not intersect every element of X^t_ℓ. We define an increasing function $C(t) : \omega \longrightarrow \omega$ such that for every $n, i \in \omega$, if $d(n) \leq i < d(n+1)$ then $C(t)(n) \setminus B_n$ contains an element of X^t_i. The rest of the proof is similar as in the case of $\diamondsuit(\mathfrak{b})$. ⊣

It is known that $\diamondsuit(\mathfrak{b})$ holds in the Cohen model (see [33]) so there are Shelah-Steprāns MAD families in this model but as we saw earlier, there is no strongly tight MAD family, so being Shelah-Steprāns does not imply being strongly tight. We will later see that strong tightness does not imply being Shelah-Steprāns.

4.5. **Existence under forcing axioms.** We will now prove two results: $\mathfrak{p} = \mathfrak{c}$, which is equivalent to MA(σ-centered), implies the existence of a Shelah-Steprāns MAD family and of a strongly tight MAD family. In [32] it was proved that Laflamme MAD families exist under $\mathfrak{p} = \mathfrak{c}$. The following is a strengthening:

Proposition 4.28. *If $\mathfrak{p} = \mathfrak{c}$ then every AD family of size less than $\mathfrak{c}$ can be extended to a Shelah-Steprāns MAD family.*

Proof. Let $\mathcal{A}$ be an AD family of size less than $\mathfrak{c}$ and $X = \{s_n \mid n \in \omega\} \in \left(\mathcal{I}(\mathcal{A})^{<\omega}\right)^+$. We define the forcing $\mathbb{P}$ as the set of all $p = (t_p, \mathcal{F}_p)$ where $t_p \in 2^{<\omega}$ and $\mathcal{F}_p \in [\mathcal{A}]^{<\omega}$. If $p = (t_p, \mathcal{F}_p)$ and $q = (t_q, \mathcal{F}_q)$ then $p \leq q$ if the following holds:

1. $t_q \subseteq t_p$ and $\mathcal{F}_q \subseteq \mathcal{F}_p$.
2. In case $n \in dom(t_p) \setminus dom(t_q)$ and $A \in \mathcal{F}_q$ if $t_p(n) = 1$ then $s_n \cap A = \emptyset$.

For any $n \in \omega$ and $A \in \mathcal{A}$ let $D_{n,A} \subseteq \mathbb{P}$ be the set of conditions $p = (t_p, \mathcal{F}_p)$ such that $t_p^{-1}(1)$ has size at least n and $A \in \mathcal{F}_p$. Since $X \in \left(\mathcal{I}(\mathcal{A})^{<\omega}\right)^+$ each $D_{n,A}$ is open dense. Clearly $\mathbb{P}$ is σ-centered and since $\mathcal{A}$ has size less than $\mathfrak{p}$ we can then force and find $Y \in [X]^\omega$ such that $\bigcup Y$ is almost disjoint with every element of $\mathcal{A}$. ⊣

We prove a similar result for strongly tight MAD families:

Lemma 4.29. *Let $\mathcal{A}$ be an AD family of size less than $\mathfrak{p}$. Let $\{X_n \mid n \in \omega\}$ be a family of infinite subsets of ω such that for every $A \in \mathcal{I}(\mathcal{A})$ the set $\{n \mid X_n \subseteq^* A\}$ is finite. Then there is $B \in \mathcal{A}^{\perp}$ such that $B \cap X_n \neq \emptyset$ for every $n \in \omega$.*

Proof. We may assume that for every $n \in \omega$ there is $A_n \in \mathcal{A}$ such that $X_n \subseteq A_n$ (note that if $A \in \mathcal{A}$ then the set $\{n \mid A_n = A\}$ is finite). Let $\mathcal{B} = \{A_n \mid n \in \omega\}$ and $\mathcal{D} = \mathcal{A} \setminus \mathcal{B}$. We now define the forcing $\mathbb{P}$ whose elements are sets of the form $p = (s_p, F_p, G_p)$ with the following properties:

(1) $s_p \in \omega^{<\omega}$, $F_p \in [\mathcal{D}]^{<\omega}$ and $G_p \in [\mathcal{B}]^{<\omega}$.
(2) If $i \in dom(s_p)$ then $s_p(i) \in X_i$.

For $p, q \in \mathbb{P}$ we let $p \leq q$ if the following conditions hold:

(1) $s_q \subseteq s_p$, $F_q \subseteq F_p$ and $G_q \subseteq G_p$.
(2) For every $i \in dom(s_p) \setminus dom(s_q)$ the following holds:
  (a) $s_p(i) \notin \bigcup F_q$.
  (b) If $B \in G_q$ and $A_i \neq B$ then $s_p(i) \notin B$.

It is easy to see that $\mathbb{P}$ is a σ-centered forcing and adds a set almost disjoint with $\mathcal{A}$ that intersects every X_n. Since $\mathcal{A}$ has size less than $\mathfrak{p}$, the result follows. $\dashv$

We conclude:

Proposition 4.30. *If $\mathfrak{p} = \mathfrak{c}$ then every AD family of size less than $\mathfrak{c}$ can be extended to a strongly tight MAD family.*

We strongly conjecture the following has a positive answer:

Problem 4.31. *Does $\mathfrak{p} = \mathfrak{c}$ imply the existence of a raving MAD family?*

4.6. **Non-implications.** Under CH we provide a number of examples for MAD families satisfying some of the properties of Subsection 4.1 while failing others.
We will now show that (consistently) strong tightness does not imply being Laflamme or random indestructible. Recall that the summable ideal is defined as $\mathcal{J}_{1/n} = \{A \subseteq \omega \mid \sum\limits_{n \in A} \frac{1}{n+1} < \omega\}$. We start with the following lemma:

Lemma 4.32. *Let $\mathcal{A}$ be a countable AD family contained in the summable ideal. Let $X = \{X_n \mid n \in \omega\} \subseteq [\omega]^{\omega}$ such that all X_n are contained in some member of $\mathcal{A}$ and $\{n \mid X_n \subseteq A\}$ is finite for all $A \in \mathcal{A}$. Then there is $D \in \mathcal{A}^{\perp} \cap \mathcal{J}_{1/n}$ such that $D \cap X_n \neq \emptyset$ for every $n \in \omega$.*

Proof. Let $\mathcal{A} = \{A_n \mid n \in \omega\}$. For each $n \in \omega$ we define $F_n = \{X_i \mid X_i \subseteq A_n\}$. We construct a sequence of finite sets $\{s_n \mid n \in \omega\} \subseteq [\omega]^{<\omega}$ such that:

(1) $\max(s_n) < \min(s_{n+1})$.
(2) $\sum\limits_{i \in s_n} \frac{1}{1+i} < \frac{1}{2^{n+1}}$.
(3) s_n has non empty intersection with every element of F_n.
(4) If $m < n$ then s_n is disjoint from A_m.

Assume we are at step n, and let r be such that $F_n = \{X_{n_1}, ..., X_{n_r}\}$. Find m such that $\frac{r}{1+m} < \frac{1}{2^{n+1}}$ and $s_i \subseteq m$ for every $i < n$. For every $i \leq r$ we choose $k_i > m$ such that $k_i \in X_{n_i} \setminus \bigcup\limits_{j<n} A_j$ and let $s_n = \{k_i \mid i \leq r\}$. It is easy to see that $D = \bigcup\limits_{n \in \omega} s_n$ has the desired properties. $\dashv$

In [16] it was proved that random forcing destroys the summable ideal (in fact, $\mathcal{J}_{1/n} \leq_K tr(\mathcal{N})$). We therefore conclude, using Lemma 4.2:

Proposition 4.33 (CH)**.** *There is a strongly tight MAD family contained in the summable ideal $\mathcal{J}_{1/n}$ (in particular, it is random destructible and not Laflamme).*

Note that the $\mathbb{P}_{\mathsf{MAD}}(\mathcal{J}_{1/n})$-generic MAD family has all these properties.

⋆⋆⋆

We saw that every fin×fin-like MAD family is Cohen indestructible (Proposition 4.13). However, we will now show that fin×fin-like does not imply tightness (so in particular, it is a weaker notion than Shelah-Steprāns).

Proposition 4.34 (CH)**.** *There is a fin×fin-like MAD family that is not tight.*

Proof. Let $\{\mathcal{I}_\alpha \mid \omega \leq \alpha < \omega_1\}$ be an enumeration of all analytic ideals $\mathcal{I}$ with $\mathsf{fin}\times\mathsf{fin} \not\leq_K \mathcal{I}$ and let $\{X_n \mid n \in \omega\}$ be a partition of ω into infinite sets. We will recursively construct an AD family $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$ such that for every α the following conditions hold:

1. $\{A_n \mid n \in \omega\}$ is a partition of ω refining $\{X_n \mid n \in \omega\}$ and every X_n contains infinitely many of the A_m.
2. There is $\xi \leq \alpha$ such that $A_\xi \notin \mathcal{I}_\alpha$.
3. If $B \in \mathcal{I}(\mathcal{A})$ then there is $n \in \omega$ such that $B \cap X_n$ is finite.

Note that by the comment after the proof of Lemma 4.7, $\mathcal{A}$ will indeed be fin×fin-like while the X_n witness the failure of tightness.

Let $\mathcal{A}_\alpha = \{A_\xi \mid \xi < \alpha\}$ and assume $\mathcal{A}_\alpha \subseteq \mathcal{I}_\alpha$. Let $\alpha = \{\alpha_n \mid n \in \omega\}$ and define $L_n = A_{\alpha_0} \cup ... \cup A_{\alpha_n}$. Define $E_n = \{m \mid |L_n \cap X_m| < \omega\}$ and note that $E = \langle E_n \rangle_{n\in\omega}$ is a decreasing sequence of infinite sets. Find a pseudointersection D of E such that $\omega \setminus D$ also contains a pseudointersection of E. Define $T_0 = \bigcup_{n\in D} X_n$ and $T_1 = \bigcup_{n\notin D} X_n$. Since $\mathsf{fin}\times\mathsf{fin} \not\leq_K \mathcal{I}_\alpha$ we know that either $\mathsf{fin}\times\mathsf{fin} \not\leq_K \mathcal{I}_\alpha \restriction T_0$ or $\mathsf{fin}\times\mathsf{fin} \not\leq_K \mathcal{I}_\alpha \restriction T_1$. First assume $\mathsf{fin}\times\mathsf{fin} \not\leq_K \mathcal{I}_\alpha \restriction T_0$. Then we can choose $A_\alpha \in (\mathcal{I}_\alpha \restriction T_0)^+$ that is almost disjoint with $\mathcal{A}_\alpha \restriction T_0$ which implies it is AD with $\mathcal{A}_\alpha$. We now need to prove that for every $n < \omega$ there is X_m such that $(L_n \cup A_\alpha) \cap X_m$ is finite. Since $\omega \setminus D$ contains a pseudointersection of E there is $m \in E_n \setminus D$ and then both L_n and A_α are almost disjoint with X_m. The other case is similar. ⊣

⋆⋆⋆

Recall that the density zero ideal is defined as $\mathcal{Z} = \{A \subseteq \omega \mid \lim \frac{|A\cap 2^n|}{2^n} = 0\}$. $\mathcal{Z}$ is not Katětov below any F_σ-ideal. Thus, from Lemma 4.15, we obtain:

Corollary 4.35. *Let $\mathcal{A}$ be a countable AD family contained in $\mathcal{Z}$. If $\mathcal{I}$ is an F_σ-ideal and $f : \omega \longrightarrow \omega$ then there is a countable AD family $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{Z}$ and there is $B \in \mathcal{I}(\mathcal{B})$ such that $f^{-1}(B) \in \mathcal{I}^+$.*

Using a suitable bookkeeping device we conclude:

Proposition 4.36 (CH)**.** *There is a Laflamme MAD contained in $\mathcal{Z}$. In particular, it is not a not-P MAD family. Additionally this MAD family is strongly tight and random indestructible.*

Indeed, the $\mathbb{P}_{\mathsf{MAD}}(\mathcal{Z})$-generic MAD family has all these properties. To see strong tightness, use Lemma 4.32 and $\mathcal{J}_{1/n} \subseteq \mathcal{Z}$. Since $\mathcal{Z} \not\leq_K tr(\mathcal{N})$, random indestructibility follows from Corollary 4.16. We shall come back to this generic object in the next section (Theorem 5.13).

5. Destructibility by forcing

In [41] Shelah constructed models of $\mathfrak{d} < \mathfrak{a}$ (see also [5]). In these models, $\mathfrak{d}$ is bigger than ω_1. It is an old question of Roitman whether $\mathfrak{d} = \omega_1$ implies $\mathfrak{a} = \omega_1$. Even the following question of the first and last authors is still open:

Problem 5.1 (Brendle, Raghavan)**.** *Does $\mathfrak{b} = \mathfrak{s} = \omega_1$ imply $\mathfrak{a} = \omega_1$?*

Constructing models of $\mathfrak{b} < \mathfrak{a}$ is much easier than constructing models of $\mathfrak{d} < \mathfrak{a}$. However, all the known models of $\mathfrak{b} = \omega_1 < \mathfrak{a}$ require diagonalizing an ultrafilter, which increases the splitting number (see [40], [4], [6] and [7]). Problem 5.1 is related to the following: *Assuming* CH, *can every* MAD *family be destroyed by a proper forcing that does not add dominating or unsplit reals?* Recall that Shelah-Steprāns MAD families $\mathcal{A}$ are indestructible for many definable forcings that do not add dominating reals. Perhaps surprisingly, such families can be destroyed by forcings that do not add dominating reals or unsplit reals. In fact, we will see that the Mathias forcing associated with $\mathcal{I}(\mathcal{A})$ has these properties.

5.1. **Destroying Hurewicz ideals.** Recall that for Hurewicz ideals $\mathcal{I}$, Mathias forcing $\mathbb{M}(\mathcal{I})$ preserves unbounded families from the ground model. We now proceed to strengthen this.

We need the following notions (see [7, Definition 31] for a notion similar to item 2):

Definition 5.2. *(1) Let* $P = \{s_n \mid n \in \omega\} \subseteq [\omega]^{<\omega}$ *be a collection of finite disjoint sets and* $S \in [\omega]^{\omega}$. *We say that* S block splits P *if both of the sets* $\{n \mid s_n \subseteq S\}$ *and* $\{s_n \mid s_n \cap S = \emptyset\}$ *are infinite.*

(2) We say that $\mathcal{S} = \{S_\alpha \mid \alpha \in \omega_1\} \subseteq [\omega]^{\omega}$ *is a* tail block-splitting family *if for every infinite set* P *of finite disjoint subsets of* ω *there is* $\alpha < \omega_1$ *such that* S_γ *block splits* P *for every* $\gamma > \alpha$.

It is easy to see that tail block splitting families exist if $\mathfrak{d} = \omega_1$ (see also [7, Observation 34]) and tail block splitting families are splitting families. We say that a forcing $\mathbb{P}$ preserves a tail block-splitting family if it remains tail block-splitting after forcing with $\mathbb{P}$.

Proposition 5.3. *Let* $\mathcal{I}$ *be a Hurewicz ideal. If* $\mathcal{S} = \{S_\alpha \mid \alpha \in \omega_1\} \subseteq [\omega]^{\omega}$ *is a tail block-splitting family then* $\mathbb{M}(\mathcal{I})$ *preserves* $\mathcal{S}$ *as a tail block-splitting family.*

Proof. Let $\mathcal{I}$ be a Hurewicz ideal and $\mathcal{S}$ a tail block-splitting family. Let $\dot{P} = \{\dot{p}_n \mid n \in \omega\}$ be a name for an infinite set of pairwise disjoint finite subsets of ω, we may assume $\dot{p}_n$ is forced to be disjoint from n. For every $s \in [\omega]^{<\omega}$ and $m \in \omega$ we define $X_m(s)$ as the set of all $t \in [\omega]^{<\omega}$ such that $\max(s) < \min(t)$ and there are $F_{(t,m,s)} \in [\omega]^{<\omega}$ and $B \in \mathcal{I}$ such that $(s \cup t, B) \Vdash$ "$\dot{p}_m = F_{(t,m,s)}$". It is easy to see that $\{X_m(s) \mid m \in \omega\} \subseteq (\mathcal{I}^{<\omega})^+$ and since $\mathcal{I}$ is Hurewicz, we may find $Y_m(s) \in [X_m(s)]^{<\omega}$ such that if $W \in [\omega]^{\omega}$ then $\bigcup_{m \in W} Y_m(s) \in (\mathcal{I}^{<\omega})^+$. Let $Z_m(s) = \bigcup_{t \in Y_m(s)} F_{(t,m,s)}$. For every $s \in [\omega]^{<\omega}$ we can then find $D(s) \in [\omega]^{\omega}$ such that $R(s) = \{Z_m(s) \mid m \in D(s)\}$ is pairwise disjoint.

Since $\mathcal{S}$ is tail block-splitting, we can find α such that if $\gamma > \alpha$ then S_γ block splits $R(s)$ for every $s \in [\omega]^{<\omega}$. We claim that in this case, S_γ is forced to block split $\dot{P}$. If this was not the case, we could find $(s, A) \in \mathbb{M}(\mathcal{I})$ and $n \in \omega$ such that either $(s, A) \Vdash$ "$\bigcup\{\dot{p}_m \mid \dot{p}_m \subseteq S_\gamma\} \subseteq n$" or $(s, A) \Vdash$ "$\bigcup\{\dot{p}_m \mid \dot{p}_m \cap S_\gamma = \emptyset\} \subseteq n$". Assume the first case holds (the other one is similar). Since S_γ block splits $R(s)$, we know that the set $W = \{m > n \mid Z_m(s) \subseteq S_\gamma\}$ is infinite. Since $\bigcup_{m \in W} Y_m(s) \in (\mathcal{I}^{<\omega})^+$, there is $m \in W$ and $t \in Y_m(s)$ such that $t \cap A = \emptyset$. We know there is $B \in \mathcal{I}$ such that $(s \cup t, B) \Vdash$ "$\dot{p}_m = F_{(t,m,s)}$". Since $t \cap A = \emptyset$, we have $(s \cup t, A \cup B) \leq (s, A)$. But $(s \cup t, A \cup B)$ forces that $\dot{p}_m$ is a subset of S_γ, which is a contradiction. We therefore conclude that $\mathcal{S}$ remains being a tail block-splitting family. ⊣

In particular, if V is a model of CH and $\mathcal{I}$ is a Hurewicz ideal, then $\mathbb{M}(\mathcal{I})$ preserves $V \cap [\omega]^{\omega}$ as a splitting family (this result has also been noted by Lyubomyr Zdomskyy). Since Hurewicz ideals are Canjar ideals, we conclude the following:

Corollary 5.4. *If $\mathcal{A}$ is Shelah-Steprāns then $\mathcal{A}$ can be destroyed with a ccc forcing that does not add dominating nor unsplit reals.*

In fact, such forcings can be iterated without adding unsplit reals, as the following result shows:

Proposition 5.5. *Let $\langle \mathbb{P}_{\alpha}, \dot{\mathbb{Q}}_{\alpha} \mid \alpha < \delta \rangle$ be a finite support iteration of ccc forcings. If $\mathbb{P}_{\alpha}$ forces that $\dot{\mathbb{Q}}_{\alpha}$ preserves tail block-splitting families, then $\mathbb{P}_{\delta}$ preserves tail block-splitting families.*

Proof. We prove the result by induction on δ. The cases where $\delta = 0$, δ is a successor ordinal or δ has uncountable cofinality are trivial, so we assume δ is a limit ordinal of countable cofinality. Fix an increasing sequence $\langle \delta_n \rangle_{n \in \omega}$ such that $\delta = \bigcup \delta_n$. Let $\mathcal{S} = \{S_{\alpha} \mid \alpha \in \omega_1\}$ be a tail block-splitting family and let $\dot{P} = \{\dot{s}_i \mid i \in \omega\}$ be a $\mathbb{P}_{\delta}$-name for a collection of finite disjoint sets. For every $n \in \omega$, we define a $\mathbb{P}_{\delta_n}$-name $\dot{P}(n) = \{\dot{s}_i(n) \mid i \in \omega\}$ as follows:

Assume $G_n \subseteq \mathbb{P}_{\delta_n}$ is a generic filter. In $V[G_n]$ we find a family $P(n) = \{s_i(n) \mid i \in \omega\}$ with the following properties:

(1) $P(n)$ is a family of pairwise disjoint sets.
(2) For every $i \in \omega$, there is $p \in \mathbb{P}_{\delta}/G_n$ such that $p \Vdash_{\mathbb{P}_{\delta}/G_n}$ "$\dot{s}_i = s_i(n)$" (where $\mathbb{P}_{\delta}/G_n$ denotes the quotient forcing).

Let $\dot{s}_i(n)$ be a $\mathbb{P}_{\delta_n}$-name for $s_i(n)$. Since $\mathbb{P}_{\delta_n}$ preserves tail block-splitting families, there is a $\mathbb{P}_{\delta_n}$-name $\dot{\alpha}_n$ for a countable ordinal such that $1_{\mathbb{P}_{\delta_n}}$ forces that if β is bigger that $\dot{\alpha}_n$ then S_{β} tail block-splits $\dot{P}(n)$. Since each $\mathbb{P}_{\delta_n}$ has the countable chain condition, we can find $\alpha < \omega_1$ such that $1_{\mathbb{P}_{\delta_n}} \Vdash$ "$\dot{\alpha}_n < \alpha$" for all n. We claim that if $\beta > \alpha$, then S_{β} is forced to block-split $\dot{P}$.

Assume this is not the case, so there are $m \in \omega$, $\beta > \alpha$, $p \in \mathbb{P}_{\delta}$ such that either $p \Vdash_{\mathbb{P}_{\delta}}$ "$\bigcup \{\dot{s}_i \mid \dot{s}_i \subseteq S_{\beta}\} \subseteq m$" or $p \Vdash_{\mathbb{P}_{\delta}}$ "$\bigcup \{\dot{s}_i \mid \dot{s}_i \cap S_{\beta} = \emptyset\} \subseteq m$". We will assume that $p \Vdash_{\mathbb{P}_{\delta}}$ "$\bigcup \{\dot{s}_i \mid \dot{s}_i \subseteq S_{\beta}\} \subseteq m$" (the other case is similar). Let $n \in \omega$ such that $p \in \mathbb{P}_{\delta_n}$. Since S_{β} is forced to block-split $\dot{P}(n)$, we can find $q \leq p$ and $j \in \omega$ such that $q \Vdash_{\mathbb{P}_{\delta_n}}$ "$\dot{s}_j(n) \nsubseteq m \wedge \dot{s}_j(n) \subseteq S_{\beta}$". Then there is a $\mathbb{P}_{\delta_n}$-name $\dot{r} \in \mathbb{P}_{\delta}$ such that $q \Vdash_{\mathbb{P}_{\delta_n}}$ "$\dot{r} \in \mathbb{P}_{\delta}/\dot{G}_n$ and $\dot{r} \Vdash_{\mathbb{P}_{\delta}/\dot{G}_n} \dot{s}_j(n) = \dot{s}_j$". Therefore, we can find $r_1 \leq q$ in $\mathbb{P}_{\delta}$ such that $r_1 \Vdash_{\mathbb{P}_{\delta}}$ "$\dot{s}_j \nsubseteq m \wedge \dot{s}_j \subseteq S_{\beta}$", which is a contradiction. $\dashv$

By iterating the Mathias forcing of all Hurewicz ideals, we obtain:

Theorem 5.6. *There is a model in which the following statements hold:*

(1) $\mathfrak{c} = \omega_2$.
(2) $\mathfrak{b} = \mathfrak{s} = \omega_1$.
(3) No MAD family of size $\aleph_1$ can be extended to a Hurewicz ideal.

We do not know the value of $\mathfrak{a}$ in the previous model. Naturally, if every MAD family could be extended to a Hurewicz ideal (at least under CH), then we would be able to solve the Problem 5.1. Unfortunately, this may not be the case, as we will prove in the next section (Theorem 8.7).

5.2. **Variants of the Shelah-Steprāns property.** Given two non-empty finite subsets s, t of ω, we write $s < t$ if $\max(s) < \min(t)$. We say that $\mathcal{B} = \{s_n \mid n \in \omega\} \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ is a *block sequence* if $s_n < s_{n+1}$ for every $n \in \omega$. The following are natural weakenings of being Shelah-Steprāns.

Definition 5.7. *Let $\mathcal{A}$ be a MAD family.*

(1) *We say that $\mathcal{A}$ is* Shelah-Steprāns for block sequences *if for every block sequence $\mathcal{B} = \{s_n \mid n \in \omega\} \in \left(\mathcal{I}(\mathcal{A})^{<\omega}\right)^+$, there is $W \in [\omega]^\omega$ such that $\bigcup_{n\in W} s_n \in \mathcal{I}(\mathcal{A})$.*
(2) *We say that $\mathcal{A}$ is* ω-Shelah-Steprāns for block sequences *if for every sequence $\langle \mathcal{B}_n \rangle_{n\in\omega}$ of block sequences with $\mathcal{B}_n \in \left(\mathcal{I}(\mathcal{A})^{<\omega}\right)^+$, there is $C \in \mathcal{I}(\mathcal{A})$ such that for every $n \in \omega$ there are infinitely many $s \in X_n$ such that $s \subseteq C$.*

Note that by Lemma 3.11, every Shelah-Steprāns MAD family is ω-Shelah-Steprāns for block sequences, and obviously ω-Shelah-Steprāns for block sequences implies Shelah-Steprāns for block sequences.

One may wonder if Shelah-Steprāns and (ω-)Shelah-Steprāns for block sequences are different concepts. We are going to prove that Shelah-Steprāns and ω-Shelah-Steprāns for block sequences might and might not agree. We do not know whether Shelah-Steprāns for block sequences and ω-Shelah-Steprāns for block sequences are the same. First we have the following result:

Lemma 5.8. *Let $\mathcal{A}$ be a MAD family such that $|\mathcal{A}| < \mathsf{cov}(\mathcal{M})$. If $\mathcal{A}$ is Shelah-Steprāns for block sequences, then $\mathcal{A}$ is Shelah-Steprāns.*

Proof. Let $\mathcal{A}$ be a MAD family of size less than $\mathsf{cov}(\mathcal{M})$ that is Shelah-Steprāns for block sequences. Letting $X = \{s_n \mid n \in \omega\} \in \left(\mathcal{I}(\mathcal{A})^{<\omega}\right)^+$, we must show that there is $B \in \mathcal{I}(\mathcal{A})$ such that B contains infinitely many elements of X. We define the forcing notion $\mathbb{P}(X)$ as the set of all p with the following properties:

(1) There is $n_p \in \omega$ such that $p : n_p \longrightarrow X$.
(2) If $i < j < n_p$ then $\max(p(i)) < \min(p(j))$.

We order $\mathbb{P}(X)$ by inclusion. Since $\mathbb{P}(X)$ is countable, it is forcing equivalent to Cohen forcing. Note that $\mathbb{P}(X)$ adds a block sequence contained in X. Furthermore, since $|\mathcal{A}| < \mathsf{cov}(\mathcal{M})$, we can find $g : \omega \longrightarrow X$ such that $g[\omega]$ is a block sequence and for every $B \in \mathcal{I}(\mathcal{A})$ there is $n \in \omega$ such that $g(n) \cap B = \emptyset$. Since $\mathcal{A}$ is Shelah-Steprāns for block sequences, we conclude that there is $B \in \mathcal{I}(\mathcal{A})$ such that B contains infinitely many elements of X. $\dashv$

It is easy to see that MAD families that are ω-Shelah-Steprāns for block sequences are tight, so in particular, they are Cohen-indestructible. We thus obtain:

Corollary 5.9. *In the Cohen model, every MAD family that is ω-Shelah-Steprāns for block sequences is Shelah-Steprāns.*

We will see now that the conclusion of the corollary is false under the Continuum Hypothesis (Theorem 5.13).

Given $n \in \omega$, let $R_n = [2^n, 2^{n+1})$ and for every $A \subseteq \omega$ let $\varphi_n(A) = \frac{|A \cap R_n|}{2^n}$. We also define the function $\varphi_{\max} : [\omega]^{<\omega} \longrightarrow \mathbb{Q}$ where $\varphi_{\max}(A) = \max\{\varphi_n(A) \mid n \in \omega\}$.

Lemma 5.10. *Let $X = \{s_n \mid n \in \omega\}$ be a block sequence and $\mathcal{A}$ a countable AD family such that $\mathcal{A} \subseteq \mathcal{Z}$. If there is $m > 0$ such that $\frac{1}{m} \leq \varphi_{\max}(s_n)$ for every $n \in \omega$, then there is $B \in \mathcal{A}^\perp \cap \mathcal{Z}$ such that $B \cap s_n \neq \emptyset$ for every $n \in \omega$.*

Proof. For every $n \in \omega$, we choose $l_n \in \omega$ such that $\frac{1}{m} \leq \varphi_{l_n}(s_n)$. Since X is pairwise disjoint, for every $l \in \omega$ the set $\{n \mid l_n = l\}$ has size at most m. Let $\mathcal{A} = \{A_n \mid n \in \omega\}$ and define $B_n = A_0 \cup ... \cup A_n$. Fix an increasing function $f : \omega \longrightarrow \omega$ such that for every $n, i \in \omega$, if $f(n) \leq i$ then $\varphi_i(B_n) < \frac{1}{m}$. We can then find $B = \{y_n \mid n \in \omega\}$ such that for every $n \in \omega$, the following holds:

(1) $y_n \in s_n \cap R_{l_n}$.
(2) If $f(k) \leq l_n$ then $y_n \notin B_k$.

It is easy to see that $B \in \mathcal{A}^{\perp}$ and $B \cap s_n \neq \emptyset$ for every $n \in \omega$. Finally, $B \in \mathcal{Z}$ since $|B \cap R_l| \leq m$ for every $l \in \omega$. ⊣

We conclude:

Lemma 5.11. *Let $\overline{X} = \{X_n \mid n \in \omega\}$ be a countable collection of block sequences and $\mathcal{A} \in \mathbb{P}_{\mathsf{MAD}}(\mathcal{Z})$. If $\mathcal{A} \Vdash$ "$\overline{X} \subseteq (\mathcal{I}(\dot{\mathcal{A}}_{gen}(\mathcal{Z}))^{<\omega})^{+}$" then there is $\mathcal{B} \in \mathbb{P}_{\mathsf{MAD}}(\mathcal{Z})$ such that $\mathcal{A} \subseteq \mathcal{B}$ and there is $B \in \mathcal{I}(\mathcal{B})$ such that B contains infinitely many elements of each X_n.*

Proof. Let $\mathcal{A} = \{A_n \mid n \in \omega\}$ and define $B_n = A_0 \cup ... \cup A_n$. Given $n, k \in \omega$, we define $X_n(k) = \{s \in X_n \mid s \cap B_k = \emptyset\}$ and note that each $X_n(k)$ is infinite. We claim that for every $n, m, k \in \omega$, $m > 0$, there are infinitely many $s \in X_n(k)$ such that $\varphi_{\max}(s) < \frac{1}{m}$. Assuming this is not the case, there are $n, m, k \in \omega$, $m > 0$ such that $\frac{1}{m} \leq \varphi_{\max}(s)$ for almost all $s \in X_n(k)$. Let $Y = \left\{s \in X_n(k) \mid \varphi_{\max}(s) < \frac{1}{m}\right\}$ and $Z = X_n(k) \setminus Y$. By the previous lemma, there is $B \in \mathcal{A}^{\perp} \cap \mathcal{Z}$ such that B intersects every element of Z. It follows that $B \cup B_k \cup \bigcup Y$ intersects every element of X_n. Therefore, $\mathcal{A} \cup \{B\}$ is an extension of $\mathcal{A}$ forcing that X_n is not positive, which is a contradiction.

Thus we know that for every $n, m, k \in \omega$, $m > 0$ there are infinitely many $s \in X_n(k)$ such that $\varphi_{\max}(s) < \frac{1}{m}$. By an easy diagonalization argument, we can find $B \in \mathcal{A}^{\perp} \cap \mathcal{Z}$ such that B contains infinitely many elements of each X_n. ⊣

Let $Z(n, m) = \left\{s \subseteq R_m \mid \varphi_m(R_m \setminus s) < \frac{1}{2^{n+1}}\right\}$ and define $X_n = \bigcup\limits_{m \in \omega} Z(n, m)$. It is easy to see that $X_n \in \left(\mathcal{I}(\mathcal{Z})^{<\omega}\right)^{+}$.

Lemma 5.12. *Let $\mathcal{A} \in \mathbb{P}_{\mathsf{MAD}}(\mathcal{Z})$ and for every $n \in \omega$, let $Y_n \in [X_n]^{<\omega}$. There is $B \in \mathcal{A}^{\perp} \cap \mathcal{Z}$ such that $B \cap s \neq \emptyset$ for every $s \in Y_n$ and $n \in \omega$.*

Proof. Let $\mathcal{A} = \{A_n \mid n \in \omega\}$ and $B_n = A_0 \cup ... \cup A_n$. We first find an increasing function $f : \omega \longrightarrow \omega$ such that for every $n \in \omega$, the following conditions hold:

(1) $f(n)$ is of the form 2^{m_n+1} for some m_n.
(2) If $f(n) < i$ then $\varphi_i(B_n) < \frac{1}{2^{n+2}}$.
(3) If $s \in Y_n$ and $s \subseteq R_j$, then $j < f(n)$.

We now define a sequence $\langle t_n \rangle_{n \in \omega}$ such that for every $n \in \omega$, the following conditions hold:

(1) $t_n \subseteq R_n$.
(2) If $n < f(0)$ then $\varphi_n(t_n) = \frac{1}{2}$.
(3) If $f(k) \leq n < f(k+1)$ then $t_n \cap B_k = \emptyset$.
(4) If $f(k) \leq n < f(k+1)$ then $\varphi_n(t_n) = \frac{1}{2^{k+2}}$.

Letting $B = \bigcup\limits_{n \in \omega} t_n$, it is easy to see that B has the desired properties. ⊣

We thus obtain:

Theorem 5.13. *(1) $\mathbb{P}_{\mathsf{MAD}}(\mathcal{Z})$ forces that $\mathcal{A}_{gen}(\mathcal{Z})$ is an ω-Shelah-Steprāns MAD family for block sequences that is not Canjar.*
(2) The Continuum Hypothesis implies that there is a non Canjar, ω-Shelah-Steprāns MAD family for block sequences $\mathcal{A}$ such that $\mathcal{A} \subseteq \mathcal{Z}$. In particular, $\mathcal{A}$ is not Shelah-Steprāns.

To see that $\mathcal{A}$ is not Shelah-Steprāns, either use $\mathcal{A} \subseteq \mathcal{Z}$ or the fact that $\mathcal{A}$ is not Canjar, and recall that every Shelah-Steprāns MAD family is Hurewicz (Proposition 4.5) and thus Canjar. For other properties of this generic MAD family see the earlier Proposition 4.36.

⋆⋆⋆

We say that a block sequence $\mathcal{B} = \{s_n \mid n \in \omega\}$ *witnesses that* $\mathcal{A}$ *is not Shelah-Steprāns for block sequences* if $\mathcal{B} \in (\mathcal{I}(\mathcal{A})^{<\omega})^+$ and there is no $W \in [\omega]^\omega$ such that $\bigcup_{n\in W} s_n \in \mathcal{I}(\mathcal{A})$. We will say that $\mathcal{B} = \{s_n \mid n \in \omega\}$ is an *increasing block sequence* if for every $n \in \omega$, the set $\{m \mid |s_m| = n\}$ is finite.

Lemma 5.14. *Let $\mathcal{A}$ be a MAD family. If a block sequence $\mathcal{B} = \{s_n \mid n \in \omega\}$ witnesses that $\mathcal{A}$ is not Shelah-Steprāns for block sequences, then $\mathcal{B}$ is an increasing block sequence.*

Proof. Assume this is not the case. So there is $m \in \omega$ such that the set $W = \{n \mid |s_n| = m\}$ is infinite. By applying that $\mathcal{A}$ is maximal m-many times, we can find $W_0 \in [W]^\omega$ and $B \in \mathcal{I}(\mathcal{A})$ such that $\bigcup\limits_{n\in W_0} s_n \subseteq B$, which is a contradiction. ⊣

We need the following notion:

Definition 5.15. *Let $\mathcal{B} = \{s_n \mid n \in \omega\}$ be an increasing block sequence. We define the ideal $\mathcal{J}(\mathcal{B})$ as the set of all $A \subseteq \omega$ such that $\lim_n \left(\frac{|A\cap s_n|}{|s_n|}\right) = 0$.*

Note that the density zero ideal has the previous form. Given an increasing block sequence $\mathcal{B} = \{s_n \mid n \in \omega\}$ and $X \in [\omega]^\omega$, we define $\mathcal{B}_X = \{s_n \mid n \in X\}$. Note that if $\mathcal{A}$ is MAD then $\mathcal{I}(\mathcal{A})$ is meager and therefore by Talagrand's Theorem, there is an increasing interval partition $\mathcal{B} = \{s_n \mid n \in \omega\}$ such that there is no $W \in [\omega]^\omega$ with $\bigcup_{n\in W} s_n \in \mathcal{I}(\mathcal{A})$. We can now prove the following result:

Proposition 5.16. *Let $\mathcal{A}$ be a MAD family and let $\mathcal{B} = \{s_n \mid n \in \omega\}$ be such that there is no $W \in [\omega]^\omega$ with $\bigcup_{n\in W} s_n \in \mathcal{I}(\mathcal{A})$. There are $X \in [\omega]^\omega$ and $\mathcal{A}_0 \in [\mathcal{A}]^{\leq\omega}$ such that $\mathcal{A} \setminus \mathcal{A}_0 \subseteq \mathcal{J}(\mathcal{B}_X)$.*

Proof. We argue by contradiction, so assume this is not the case.

By $\mathbb{Q}^+$ we denote the set of all positive rational numbers. We will now recursively define $\langle A_\alpha, q_\alpha, X_\alpha\rangle_{\alpha\in\omega_1}$ such that for every $\alpha < \omega_1$ the following hold:

1. $A_\alpha \in \mathcal{A}$, $q_\alpha \in \mathbb{Q}^+$ and $X_\alpha \in [\omega]^\omega$.
2. If $\alpha \neq \beta$ then $A_\alpha \neq A_\beta$.
3. If $\beta < \alpha$ then $X_\alpha \subseteq^* X_\beta$.
4. If $n \in X_\alpha$, then $q_\alpha \leq \frac{|A_\alpha\cap s_n|}{|s_n|}$.

Let $\alpha < \omega_1$ and assume we have already constructed $\langle A_\xi, q_\xi, X_\xi\rangle_{\xi<\alpha}$. We will see how to find A_α, q_α and X_α. Since $\{X_\xi \mid \xi < \alpha\}$ is a $\subseteq^*$-decreasing sequence, we may find $Y \in [\omega]^\omega$ such that $Y \subseteq^* X_\xi$ for every $\xi < \alpha$. By our assumption, the set $\mathcal{C} = \mathcal{A} \setminus \mathcal{J}(\mathcal{B}_Y)$ is uncountable. Note that if $A \in \mathcal{C}$, then there is $q_A \in \mathbb{Q}^+$ such that the set $\left\{n \in Y \mid q_A \leq \frac{|A\cap s_n|}{|s_n|}\right\}$ is infinite. Since $\mathcal{C}$ is uncountable, we may find $q_\alpha \in \mathbb{Q}^+$ such that $\mathcal{C}_1 = \{A \in \mathcal{C} \mid q_A = q_\alpha\}$ is uncountable. We can then find $A_\alpha \in \mathcal{C}_1$ such that $A_\alpha \neq A_\xi$ for every $\xi < \alpha$. Finally, let $X_\alpha = \left\{n \in Y \mid q_\alpha \leq \frac{|A_\alpha\cap s_n|}{|s_n|}\right\}$. Clearly, A_α, q_α and X_α have the desired properties.

We can now find $W \in [\omega_1]^{\omega_1}$ and $q \in \mathbb{Q}^+$ such that $q_\alpha = q$ for every $\alpha \in W$. Let $m \in \omega$ such that $\frac{1}{q} < m$ and choose $\alpha_1, ..., \alpha_m \in W$. Let $X = X_{\alpha_1} \cap ... \cap X_{\alpha_m}$ and note that X is an infinite set. By construction, if $n \in X$ and $i \leq m$ then $q \leq \frac{|A_{\alpha_i}\cap s_n|}{|s_n|}$. Since $\frac{1}{q} < m$, for each $n \in X$ there must be $i_n, j_n \leq m$ such that $i_n \neq j_n$ and $A_{\alpha_{i_n}} \cap A_{\alpha_{j_n}} \cap s_n \neq \emptyset$. Since X is infinite, there are $i, j \leq m$ and $Y \in [X]^\omega$ such that $i = i_n$ and $j = j_n$ for every $n \in Y$. This implies that $A_{\alpha_{i_n}} \cap A_{\alpha_{j_n}}$ is infinite, which is a contradiction. ⊣

We also have the following:

Lemma 5.17. *If $\mathcal{B}$ is an increasing block sequence, then $\mathcal{J}(\mathcal{B}) \leq_K \mathcal{Z}$.*

Proof. It is easy to see that $\mathcal{J}(\mathcal{B})$ is a non-pathological P-ideal (see [13] or [29] for the definition of non-pathological P-ideals) and in [29] it was proved that every non-pathological P-ideal is Katětov below $\mathcal{Z}$. $\dashv$

Therefore, every MAD family is "nearly Katětov-below" $\mathcal{Z}$. By these results, it would be tempting to conjecture the following: *If $\mathcal{Z}$ can be destroyed without increasing $\mathfrak{b}$ and $\mathfrak{s}$, then every MAD family can be destroyed without increasing $\mathfrak{b}$ or $\mathfrak{s}$.* Unfortunately, it seems that the density zero ideal can not be destroyed without increasing $\mathfrak{b}$ or $\mathfrak{s}$. Recently, Raghavan showed that $\mathsf{cov}^*(\mathcal{Z}) \leq \max\{\mathfrak{b}, \mathfrak{s}(\mathfrak{pr})\}$[4] ($\mathfrak{s}(\mathfrak{pr})$ is the *promptly splitting number*, which is a cardinal invariant closely related to $\mathfrak{s}$, see [37] for more details). This improves an earlier work of Raghavan and Shelah (see [38]) where they showed that $\mathsf{cov}^*(\mathcal{Z}) \leq \mathfrak{d}$.

We know that every MAD family is contained up to a countable subfamily in an ideal $\mathcal{J}(\mathcal{B})$ (where $\mathcal{B}$ is an increasing block sequence). We will now show that (consistently) this is best possible, that is, one can not disregard the countable family in Proposition 5.16 (see Theorem 5.20).

Let $\mathcal{B} = \{P_n \mid n \in \omega\}$ be the interval partition of ω with $|P_n| = n+1$. Given $X \subseteq \omega$ and $n \in \omega$, we define $\mathcal{B}(X,n) = \{m \mid |P_m \setminus X| \leq n\}$. We will say a family $\mathcal{A}$ is $\mathcal{B}$-AD if the following conditions hold:

(1) $\mathcal{A}$ is a countable AD family.
(2) If $B \in \mathcal{I}(\mathcal{A})$ then $\mathcal{B}(B,n)$ is finite for every $n \in \omega$.
(3) If $B \in \mathcal{I}(\mathcal{A})$ then there is $n \in \omega$ such that $P_n \cap B = \emptyset$.

Note that if $\mathcal{A}$ is $\mathcal{B}$-AD then for every $B \in \mathcal{I}(\mathcal{A})$ there are, in fact, infinitely many $n \in \omega$ such that $P_n \cap B = \emptyset$ (recall every finite set is in $\mathcal{I}(\mathcal{A})$).

Lemma 5.18. *Let $\mathcal{A}$ be a $\mathcal{B}$-AD and $X \in \mathcal{A}^{\perp}$. There is $A \in [X]^{\omega}$ such that $\mathcal{A} \cup \{A\}$ is $\mathcal{B}$-AD.*

Proof. Letting $\mathcal{A} = \{A_n \mid n \in \omega\}$, we may assume that $n \in A_n$ for every $n \in \omega$. For every $n \in \omega$, we define $B_n = \bigcup \{A_i \mid i \leq \max(P_n)\}$ (note that $P_0 \cup ... \cup P_n \subseteq B_n$ and $B_n \in \mathcal{I}(\mathcal{A})$). We recursively construct a sequence $\langle (y_n, u_n, x_n) \rangle_{n\in\omega}$ with the following properties:

(1) $y_n < u_n < y_{n+1}$ for every $n \in \omega$.
(2) $\{x_n \mid n \in \omega\} \subseteq X$.
(3) $x_n \in P_{y_n}$.
(4) $x_n \notin B_n$.
(5) $y_n \notin \mathcal{B}(B_n, n+1)$.
(6) $B_n \cap P_{u_n} = \emptyset$.

Assuming we have constructed (y_i, u_i, x_i) for every $i < n$, we will see how to define (y_n, u_n, x_n). We first find $r \in \omega$ such that the following hold:

(1) $\max(P_{u_i}) < r$ for every $i < n$.
(2) $B_n \cap X \subseteq r$.
(3) $\mathcal{B}(B_n, n+1) \subseteq r$.

Since X is an infinite set, we can find y_n such that $r < \min(P_{y_n})$ and $X \cap P_{y_n} \neq \emptyset$. Choose any $x_n \in X \cap P_{y_n}$. Finally, let u_n such that $y_n < u_n$ and $B_n \cap P_{u_n} = \emptyset$.

We now define $A = \{x_n \mid n \in \omega\}$. Clearly A is almost disjoint with every element of $\mathcal{A}$ and A is an infinite subset of X. Letting $\mathcal{A}_1 = \mathcal{A} \cup \{A\}$, we need to argue that $\mathcal{A}_1$ is a $\mathcal{B}$-AD family. Letting $n \in \omega$, note that if $m > n$ then $(A \cup B_n) \cap P_{u_m} = \emptyset$,

[4] If $\mathcal{I}$ is a tall ideal, by $\mathsf{cov}^*(\mathcal{I})$ we denote the smallest size of a family $\mathcal{X} \subseteq \mathcal{I}$ such that for every $A \in [\omega]^{\omega}$ there is $X \in \mathcal{X}$ such that $A \cap X \in [\omega]^{\omega}$.

so $A \cup B_n$ is disjoint with infinitely many elements of $\mathcal{B}$. Finally, note that if $m > n$ then $y_m \notin \mathcal{B}(A \cup B_n, n)$. $\dashv$

We can now prove:

Lemma 5.19. *Let $\mathcal{A}$ be $\mathcal{B}$-AD. If $f : \omega \longrightarrow \omega$ is a Katětov-morphism from $\mathcal{Z}$ to $\mathcal{I}(\mathcal{A})$ then there is $A \in \mathcal{A}^{\perp}$ such that $\mathcal{A}_1 = \mathcal{A} \cup \{A\}$ is $\mathcal{B}$-AD and f is no longer a Katětov-morphism from $\mathcal{Z}$ to $\mathcal{I}(\mathcal{A}_1)$.*

Proof. As before, let $\mathcal{A} = \{A_n \mid n \in \omega\}$ (we assume again that $n \in A_n$ for every $n \in \omega$) and for every $n \in \omega$, we define $B_n = \bigcup \{A_i \mid i \leq \max(P_n)\}$. We recursively construct a sequence $\langle (y_n, u_n, s_n) \rangle_{n \in \omega}$ with the following properties:

1. $y_n < y_{n+1}$ for every $n \in \omega$.
2. $s_n \subseteq f[R_{y_n}]$ (recall that $R_n = [2^n, 2^{n+1})$ and for every $A \subseteq \omega$ we defined $\varphi_n(A) = \frac{|A \cap R_n|}{2^n}$).
3. $\varphi_{y_n}\left(f^{-1}(s_n)\right) \geq \frac{1}{3}$.
4. $s_n \cap B_n = \emptyset$.
5. If $P_m \cap s_n \neq \emptyset$ then $m \notin \mathcal{B}(B_n, 3n)$ for every $m \in \omega$.
6. If $P_m \cap s_n \neq \emptyset$ then $n \leq |P_m \setminus (s_n \cup B_n)|$.
7. $P_{u_n} \cap B_n = \emptyset$.
8. $\max(P_{u_n}) < \min(s_{n+1}) \leq \max(s_{n+1}) < \min\left(P_{u_{n+1}}\right)$.

Assuming we have constructed the triple (y_n, u_n, s_n), we will show how to construct $(y_{n+1}, u_{n+1}, s_{n+1})$. Let $D = B_{n+1} \cup \bigcup \{P_m \mid m \in \mathcal{B}(B_{n+1}, 3(n+1))\} \cup \max(P_{u_n})$. Clearly $D \in \mathcal{I}(\mathcal{A})$, hence $f^{-1}(D) \in \mathcal{Z}$. Let $y_{n+1} > y_n$ with the property that $\varphi_{y_{n+1}}\left(f^{-1}(D)\right) < \frac{1}{3}$. Let $z = f\left[R_{y_{n+1}} \setminus f^{-1}(D)\right]$ and note that if $m \in \omega$ and $P_m \cap z \neq \emptyset$ then $m \notin \mathcal{B}(B_{n+1}, 3(n+1))$ and $\max(P_{u_n}) < \min(P_m)$. Let $K = \{m \mid P_m \cap z \neq \emptyset\}$ which clearly is a finite set. For every $m \in K$ let $t_m = P_m \setminus B_{n+1}$ and define $K_1 = \{m \in K \mid |t_m \setminus z| < n+1\}$. Note if $m \in K_1$, then $|t_m| > 3(n+1)$ hence $2(n+1) \leq |t_m \cap z|$. For $m \in K_1$ we can now choose distinct $x_0^m, ..., x_n^m, w_0^m, ..., w_n^m \in t_m \cap z$ such that $\varphi_{y_{n+1}}\left(f^{-1}(\{x_0^m, ..., x_n^m\})\right) \geq \varphi_{y_{n+1}}\left(f^{-1}(\{w_0^m, ..., w_n^m\})\right)$. Given $m \in K$, we now define the set $s_n^m = (t_m \cap z) \setminus \{w_0^m, ..., w_n^m\}$ if $m \in K_1$ and $s_n^m = t_m \cap z$ if $m \in K \setminus K_1$. Let $\overline{s}_n^m = \{w_0^m, ..., w_n^m\}$ for every $m \in K_1$ and we define $s_{n+1} = \bigcup\limits_{m \in K} s_n^m$ and $\overline{s}_{n+1} = \bigcup\limits_{m \in K_1} \overline{s}_n^m$. Note that $R_{y_{n+1}} = R_{y_{n+1}} \cap \left(f^{-1}(D) \cup f^{-1}(s_{n+1}) \cup f^{-1}(\overline{s}_{n+1})\right)$. Now, $\varphi_{y_{n+1}}\left(f^{-1}(D)\right) < \frac{1}{3}$ and $\varphi_{y_{n+1}}\left(f^{-1}(\overline{s}_{n+1})\right) \leq \varphi_{y_{n+1}}\left(f^{-1}(s_{n+1})\right)$, hence $\varphi_{y_{n+1}}\left(f^{-1}(s_{n+1})\right) \geq \frac{1}{3}$. It is easy to see that s_{n+1} has all the desired properties. Finally we choose u_{n+1} accordingly.

Letting $A = \bigcup\limits_{n \in \omega} s_n$, it is easy to see that $f^{-1}(A) \notin \mathcal{Z}$ and $\mathcal{A} \cup \{A\}$ is $\mathcal{B}$-AD. $\dashv$

With a suitable bookkeeping device, we conclude the following:

Theorem 5.20 (CH)**.** *There is a $\mathcal{Z}$-MAD family that is not Shelah-Steprāns for block sequences.*

Therefore, the countable family mentioned in Proposition 5.16 is indeed needed. Theorem 5.20 motivates the following questions:

Problem 5.21. *(1) Do $\mathcal{Z}$-MAD families exist (in ZFC)?*
(2) Is it consistent that every MAD family that is not Shelah-Steprāns (or Shelah-Steprāns for block sequences) is Katětov below $\mathcal{Z}$?

6. A general strategy for producing models with small non-meager families

A general strategy for preserving certain non-meager sets from the ground model is provided in this section. Usual methods of producing models of $\text{non}(\mathcal{M}) = \aleph_1$

with a countable support iteration rely on the preservation of stronger properties such as $\sqsubseteq_{\text{Cohen}}$. Our results here allow us to keep $\mathrm{non}(\mathcal{M}) = \aleph_1$ in countable support iterations where the iterands do not enjoy any of these additional preservation properties. The results of this section will be used in Section 7 to produce a model of $\mathrm{non}(\mathcal{M}) = \aleph_1$ where there are no block Shelah-Steprāns a.d. families of size $\aleph_1$. However, our results here are quite general and we expect they will have further applications going beyond almost disjoint families.

Definition 6.1. An *interval partition or IP* is a sequence $I = \langle i_n : n \in \omega \rangle \in \omega^\omega$ such that $i_0 = 0$ and $\forall n \in \omega\, [i_n < i_{n+1}]$.

Given an IP I and $n \in \omega$, I_n denotes $[i_n, i_{n+1}) = \{l \in \omega : i_n \leq l < i_{n+1}\}$.

The following is a slight variation of a well-known connection between eventually different reals and meagerness of the ground model. We give a proof even though the argument is similar to the arguments in Miller [31] or Bartoszynski and Judah [2].

Lemma 6.2. *Let $\mathbf{V}_0 \subseteq \mathbf{V}_1$ be transitive models of a sufficiently large fragment of* ZFC. *Assume that*

$$\forall f \in H(\aleph_0)^\omega \cap \mathbf{V}_1 \exists g \in H(\aleph_0)^\omega \cap \mathbf{V}_0 \exists^\infty n \in \omega\, [f(n) = g(n)].$$

Then the following hold:

(1) *for each $f \in H(\aleph_0)^\omega \cap \mathbf{V}_1$ and each $X \in [\omega]^\omega \cap \mathbf{V}_1$, there exists a $g \in H(\aleph_0)^\omega \cap \mathbf{V}_0$ so that*
$$\exists^\infty n \in X\, [f(n) = g(n)];$$
(2) *for each IP $I = \langle i_n : n \in \omega \rangle \in \omega^\omega \cap \mathbf{V}_1$, there is an IP $J = \langle j_n : n \in \omega \rangle \in \omega^\omega \cap \mathbf{V}_0$ so that*
$$\exists^\infty l \in \omega \exists n \in \omega\, [I_n \subseteq J_l];$$
(3) *for any $M \in \mathbf{V}_1$, if $(M \subseteq 2^\omega \text{ is a meager set})^{\mathbf{V}_1}$, then there exists $x \in 2^\omega \cap \mathbf{V}_0$ with $x \notin M$.*

Proof. For (1): working in $\mathbf{V}_1$, fix $f \in H(\aleph_0)^\omega$ and $X \in [\omega]^\omega$. Let $\langle x_n : n \in \omega \rangle$ be the strictly increasing enumeration of X. Recall that whenever $A \in H(\aleph_0)$, then $H(\aleph_0)^A \subseteq H(\aleph_0)$. For each $n \in \omega$, $\{x_i : i < n\}$, being a finite subset of ω, is a member of $H(\aleph_0)$. Therefore $f{\restriction}\{x_i : i < n\} \in H(\aleph_0)^{\{x_i : i<n\}} \subseteq H(\aleph_0)$. So we may define a function $F : \omega \to H(\aleph_0)$ by setting $F(n) = f{\restriction}\{x_i : i < n\}$, for each $n \in \omega$. By hypothesis, there is a function $G \in \mathbf{V}_0$ so that $G : \omega \to H(\aleph_0)$ and $\exists^\infty n \in \omega\, [F(n) = G(n)]$. Observe that whenever $F(n) = G(n)$, $G(n)$ is a function and $\mathrm{dom}(G(n)) \in [\omega]^n$. Thus working in $\mathbf{V}_0$, we see that the set $Y = \{n \in \omega : G(n) \text{ is a function and } \mathrm{dom}(G(n)) \in [\omega]^n\}$ is infinite. Let $\langle y_n : n \in \omega \rangle$ be the strictly increasing enumeration of Y. Define a function $e \in \omega^\omega$ by induction as follows. Let $n \in \omega$ and assume that $e(m)$ has been defined for all $m < n$. Then $|\{e(m) : m < n\}| \leq n$, while $|\mathrm{dom}(G(y_{n+1}))| = y_{n+1} \geq n + 1 > n$. So define $e(n) = \min(\mathrm{dom}(G(y_{n+1})) \setminus \{e(m) : m < n\})$. Note that e is a 1-1 function and that $e(n) \in \mathrm{dom}(G(y_{n+1}))$, which means that $G(y_{n+1})(e(n))$ is defined and is a member of $H(\aleph_0)$. Define $g : \omega \to H(\aleph_0)$ so that $g(e(n)) = G(y_{n+1})(e(n))$, for all $n \in \omega$, while $g(k) = 0$, for all $k \notin \mathrm{ran}(e)$. We check that g is as needed. We know $Z = \{n \in \omega : F(y_{n+1}) = G(y_{n+1})\}$ is infinite. If $n \in Z$, then $\mathrm{dom}(G(y_{n+1})) \subseteq X$, whence $e(n) \in X$ and $g(e(n)) = G(y_{n+1})(e(n)) = F(y_{n+1})(e(n)) = f(e(n))$. Thus $e''Z \subseteq \{k \in X : g(k) = f(k)\}$. As e is a 1-1 function, $\{k \in X : g(k) = f(k)\}$ is an infinite set.

For (2): working in $\mathbf{V}_1$, fix an IP $I = \langle i_n : n \in \omega \rangle$. Define $f : \omega \to H(\aleph_0)$ by $f(n) = i_{n+2}$, for all $n \in \omega$. By the hypothesis, there is a $g \in \mathbf{V}_0$ so that $g : \omega \to H(\aleph_0)$ and $\exists^\infty n \in \omega\, [g(n) = i_{n+2}]$. Working in $\mathbf{V}_0$, define an IP $J =$

$\langle j_l : l \in \omega \rangle$ as follows. $j_0 = 0$. Fix $l \in \omega$ and suppose that $j_l \in \omega$ is given. Define $j_{l+1} = \max(\{j_l + 1\} \cup (\{g(n) : n \leq j_l\} \cap \omega))$. Note $j_{l+1} \geq j_l + 1 > j_l$, so we have an IP. To check that it is as needed, fix any $M \in \omega$. Choose $n > j_{M+1}$ with $g(n) = i_{n+2}$. There is a unique $k \in \omega$ with $n \in J_k$, that is $j_k \leq n < j_{k+1}$. Observe $k \geq M + 1 > M$. If $i_{n+1} < j_{k+1}$, then we have $j_k \leq n \leq i_n < i_{n+1} < j_{k+1}$, implying that $I_n \subseteq J_k$. If $j_{k+1} \leq i_{n+1}$, then by the definition of j_{k+2}, we have $j_{k+1} \leq i_{n+1} < i_{n+2} \leq j_{k+2}$, implying that $I_{n+1} \subseteq J_{k+1}$. Thus we have found an $l > M$, namely either $l = k$ or $l = k + 1$, and an n', namely either $n' = n$ or $n' = n + 1$, so that $I_{n'} \subseteq J_l$.

For (3): working in $\mathbf{V}_1$, fix a meager set $M \subseteq 2^\omega$. Let $\langle F_n : n \in \omega \rangle$ be a sequence of closed nowhere dense subsets of 2^ω such that $\forall n \in \omega\, [F_n \subseteq F_{n+1}]$ and $M \subseteq \bigcup_{n\in\omega} F_n$. Build an IP $I = \langle i_n : n \in \omega \rangle$ and a sequence $\langle \tau_n : n \in \omega \rangle$ as follows. Put $i_0 = 0$. Let $n \in \omega$ and assume that $i_n \in \omega$ is given and that τ_m has been defined for all $m < n$. Find $i_{n+1} > i_n$ and a function $\tau_n : [i_n, i_{n+1}) \to 2$ such that for each $\sigma : i_n \to 2$, $[\sigma \cup \tau_n] \cap F_n = 0$. This is possible because F_n is closed nowhere dense. Note that $y = \bigcup_{n\in\omega} \tau_n : \omega \to 2$. Using (2), find an IP $J = \langle j_l : l \in \omega \rangle \in \mathbf{V}_0$ with the property that $\exists^\infty l \in \omega \exists n \in \omega\, [I_n \subseteq J_l]$. In $\mathbf{V}_1$, define a function $F : \omega \to H(\aleph_0)$ by setting $F(l) = y\restriction [j_l, j_{l+1})$, for all $l \in \omega$. Note that if $I_n \subseteq J_l$, then $\tau_n = y\restriction I_n \subseteq y\restriction J_l = F(l)$. Let $X = \{l \in \omega : \exists n \in \omega\, [I_n \subseteq J_l]\}$. Applying (1), find $G \in \mathbf{V}_0$ such that $G : \omega \to H(\aleph_0)$ and $\exists^\infty l \in X\, [G(l) = F(l)]$. Working in $\mathbf{V}_0$, define $x : \omega \to 2$ such that for each $l \in \omega$, if $G(l) : J_l \to 2$, then $x\restriction J_l = G(l)$, while if not, then $x\restriction J_l$ is constantly 0. Suppose $l \in X$ and that $G(l) = F(l)$. Choose $n \in \omega$ with $I_n \subseteq J_l$. Let $\sigma = x\restriction i_n : i_n \to 2$. By the choice of τ_n, $[\sigma \cup \tau_n] \cap F_n = 0$. Since $F(l) : J_l \to 2$, $I_n \subseteq J_l$, and $G(l) = F(l)$, $\tau_n \subseteq F(l) = G(l) \subseteq x$. So $x \in [\sigma \cup \tau_n]$, whence $x \notin F_n$. Thus we conclude that $\exists^\infty n \in \omega\, [x \notin F_n]$. As the F_n are increasing, $x \notin \bigcup_{n\in\omega} F_n$, and so $x \notin M$. ⊣

Definition 6.3. $S : \omega \to H(\aleph_0)$ is called a *slalom* if $\forall n \in \omega \left[|S(n)| \leq (n + 1)2^{n+1}\right]$. $S : \omega \to H(\aleph_0)$ is called a *small slalom* if $\forall n \in \omega \left[|S(n)| \leq 2^{n+1}\right]$.

Definition 6.4. Let $f \in H(\aleph_0)^\omega$. For $n \in \omega$, $A_n = \{k \leq n : f(k)$ is a function $\wedge$ $n \in \mathrm{dom}(f(k)) \wedge |f(k)(n)| \leq 2^{n+1}\}$.

Define $S_f(n) = \bigcup\{f(k)(n) : k \in A_n\}$, $\forall n \in \omega$. We observe that

$$\forall n \in \omega \left[S_f(n) \in H(\aleph_0) \text{ and } |S_f(n)| \leq (n + 1)2^{n+1}\right].$$

Thus $S_f : \omega \to H(\aleph_0)$ and S_f is a slalom.

Lemma 6.5. *Assume $\mathcal{F} \subseteq H(\aleph_0)^\omega$ is such that $\forall f \in \mathcal{F}\, [S_f \in \mathcal{F}]$. Then if $\forall f \in H(\aleph_0)^\omega \exists g \in \mathcal{F} \exists^\infty n \in \omega\, [f(n) = g(n)]$, then for any small slalom $S : \omega \to H(\aleph_0)$ and any sequence $\langle X_l : l \in \omega \rangle \subseteq [\omega]^\omega$, there is a slalom $T : \omega \to H(\aleph_0)$ so that $T \in \mathcal{F}$ and $\forall l \in \omega \exists^\infty n \in X_l\, [S(n) \subseteq T(n)]$.*

Proof. Let $I = \langle i_k : k \in \omega \rangle$ be an IP such that $\forall k \in \omega \forall l \leq k\, [X_l \cap I_k \neq \emptyset]$. Define $f : \omega \to H(\aleph_0)$ by $f(k) = S\restriction I_k$, for all $k \in \omega$. Note that for each $k \in \omega$, $H(\aleph_0)^{I_k} \subseteq H(\aleph_0)$, and so $f : \omega \to H(\aleph_0)$. By hypothesis, fix $g \in \mathcal{F}$ so that $\exists^\infty k \in \omega\, [f(k) = g(k)]$. Let $T = S_g$, which by definition means that T is a slalom. Also by hypothesis, $T \in \mathcal{F}$. To see that T is as required, fix $l \in \omega$. Let $M \in \omega$ be given. Let $K = \max\{M, l\} \in \omega$. Choose $k > K$ with $f(k) = g(k)$. Since $l \leq K < k$, $X_l \cap I_k \neq \emptyset$. Choose $n \in X_l \cap I_k$. Thus $n \in X_l$ and $M \leq K < k \leq i_k \leq n$. As $g(k) = f(k)$, we have that $k \in A_{g,n}$, whence $S(n) = g(k)(n) \subseteq S_g(n) = T(n)$. This is as needed because $n \in X_l$ and $n > M$. ⊣

Definition 6.6. Let $<_{\mathtt{wo}}$ be a well-ordering of $H(\aleph_0)$. For any $A \in H(\aleph_0)$, $\langle A, <_{\mathtt{wo}} \rangle$ is a finite well-order, and therefore there is a unique function $e_{A,<_{\mathtt{wo}}} : |A| \to A$ which is an order isomorphism from $\langle |A|, \in \rangle$ to $\langle A, <_{\texttt{wo}} \rangle$. Define an IP $I = \langle i_n : n \in \omega \rangle$ as follows. $i_0 = 0$. Given $i_n \in \omega$, $i_{n+1} = i_n + (n+1)2^{n+1}$.

Now suppose $S : \omega \to H(\aleph_0)$ is a slalom. Define $f_{S,<_{\texttt{wo}}} : \omega \to H(\aleph_0)$ as follows. Given $x \in \omega$, there exist unique $n, j \in \omega$ satisfying $i_n \le x < i_{n+1}$, $j < (n+1)2^{n+1}$, and $x = i_n + j$. If $j \ge |S(n)|$, then $f_{S,<_{\texttt{wo}}}(x) = 0 \in H(\aleph_0)$. If $j < |S(n)|$, then $e_{S(n),<_{\texttt{wo}}}(j) \in H(\aleph_0)$. If $e_{S(n),<_{\texttt{wo}}}(j)$ is a function and $\mathrm{dom}(e_{S(n),<_{\texttt{wo}}}(j)) = [i_n, i_{n+1})$, then $f_{S,<_{\texttt{wo}}}(x) = e_{S(n),<_{\texttt{wo}}}(j)(x) \in H(\aleph_0)$. Otherwise, $f_{S,<_{\texttt{wo}}}(x) = 0 \in H(\aleph_0)$.

The point of this somewhat cumbersome definition is the following lemma.

Lemma 6.7. *Assume $\mathcal{F} \subseteq H(\aleph_0)^\omega$ has the property that for every slalom $S : \omega \to H(\aleph_0)$, if $S \in \mathcal{F}$, then $f_{S,<_{\texttt{wo}}} \in \mathcal{F}$. Then if for every $f \in H(\aleph_0)^\omega$ there is an $S \in \mathcal{F}$ such that $S : \omega \to H(\aleph_0)$ is a slalom and $\exists^\infty k \in \omega\, [f(k) \in S(k)]$, then $\forall f \in H(\aleph_0)^\omega \exists g \in \mathcal{F} \exists^\infty k \in \omega\, [f(k) = g(k)]$.*

Proof. Let $f : \omega \to H(\aleph_0)$ be given. Define the IP $I = \langle i_n : n \in \omega \rangle$ as follows: $i_0 = 0$; given $i_n \in \omega$, $i_{n+1} = i_n + (n+1)2^{n+1}$. Define $F : \omega \to H(\aleph_0)$ by setting $F(k) = f \restriction I_k$. By hypothesis, find a slalom $S \in \mathcal{F}$ such that $\exists^\infty k \in \omega\, [F(k) \in S(k)]$. Consider any $k \in \omega$ such that $F(k) \in S(k)$. There exists a unique $j < |S(k)| \le (k+1)2^{k+1}$ with $e_{S(k),<_{\texttt{wo}}}(j) = F(k)$. Let $x = i_k + j$. Then $i_k \le x < i_{k+1}$ and by definition $f_{S,<_{\texttt{wo}}}(x) = e_{S(k),<_{\texttt{wo}}}(j)(x) = F(k)(x) = f(x)$. Thus for each $k \in \omega$ such that $F(k) \in S(k)$, there exists $x_k \in I_k$ such that $f_{S,<_{\texttt{wo}}}(x_k) = f(x_k)$. Since there are infinitely many such k and since $I_k \cap I_{k'} = \emptyset$ whenever $k \ne k'$, $\exists^\infty x \in \omega\, [f_{S,<_{\texttt{wo}}}(x) = f(x)]$. We are done because $f_{S,<_{\texttt{wo}}} \in \mathcal{F}$ by hypothesis. $\dashv$

Definition 6.8. Let $<_{\texttt{wo}}$ be fixed. A family $\mathcal{F} \subseteq H(\aleph_0)^\omega$ is called *well-closed w.r.t.* $<_{\texttt{wo}}$ if it satisfies the following:

1. $\forall f \in \mathcal{F}\, [S_f \in \mathcal{F}]$;
2. for every $S \in \mathcal{F}$, if $S : \omega \to H(\aleph_0)$ is a slalom, then $f_{S,<_{\texttt{wo}}} \in \mathcal{F}$;
3. for every set $\{f_n : n \in \omega\} \subseteq \mathcal{F}$, there is a slalom $S \in \mathcal{F}$ so that $\forall n \in \omega \forall^\infty k \in \omega\, [f_n(k) \in S(k)]$.

Remark 6.9. Suppose $\mathbf{V}_0 \subseteq \mathbf{V}_1$ are transitive models of a sufficiently large fragment of ZFC with $<_{\texttt{wo}} \in \mathbf{V}_0$. Then for any $A \in H(\aleph_0)$, the map $e_{A,<_{\texttt{wo}}} : |A| \to A$ is the same whether it is calculated in $\mathbf{V}_0$ or in $\mathbf{V}_1$. Also, if $S \in \mathbf{V}_0$, then S is a slalom in $\mathbf{V}_0$ if and only if S is a slalom in $\mathbf{V}_1$, and the computation of $f_{S,<_{\texttt{wo}}}$ does not change. Similarly, for any $f \in H(\aleph_0)^\omega \cap \mathbf{V}_0$, S_f is the same when calculated in $\mathbf{V}_0$ or $\mathbf{V}_1$. So we conclude that if $\mathcal{F} \in \mathbf{V}_0$ and

$$\left(\mathcal{F} \subseteq H(\aleph_0)^\omega \text{ satisfies (1) and (2) of Definition 6.8 w.r.t. } <_{\texttt{wo}}\right)^{\mathbf{V}_0}, \text{ then}$$

$$\left(\mathcal{F} \subseteq H(\aleph_0)^\omega \text{ satisfies (1) and (2) of Definition 6.8 w.r.t. } <_{\texttt{wo}}\right)^{\mathbf{V}_1}.$$

If the models $\mathbf{V}_0 \subseteq \mathbf{V}_1$ satisfy the further condition that

$$\forall X \in \mathbf{V}_0 \forall Y \in \left([X]^{\le \aleph_0}\right)^{\mathbf{V}_1} \exists Z \in \left([X]^{\le \aleph_0}\right)^{\mathbf{V}_0} [Y \subseteq Z],$$

then for any $\mathcal{F} \in \mathbf{V}_0$, if

$$\left(\mathcal{F} \text{ is well-closed w.r.t. } <_{\texttt{wo}}\right)^{\mathbf{V}_0}, \text{ then}$$

$$\left(\mathcal{F} \text{ is well-closed w.r.t. } <_{\texttt{wo}}\right)^{\mathbf{V}_1}.$$

In particular, this holds whenever $\mathbf{V}_1$ is a forcing extension of $\mathbf{V}_0$ by a proper poset.

Remark 6.9 will used several times in what follows.

Definition 6.10. $\mathcal{F} \subseteq H(\aleph_0)^\omega$ is called *big* if

$$\forall f \in H(\aleph_0)^\omega \exists g \in \mathcal{F} \exists^\infty k \in \omega\, [f(k) = g(k)].$$

Lemma 6.11. *Suppose $\mathcal{F} \subseteq H(\aleph_0)^\omega$ is big. Let $\mathbb{P}$ be a forcing such that $\Vdash_{\mathbb{P}} \mathcal{F}$ is big. Let $\mathring{\mathbb{Q}}$ be a $\mathbb{P}$-name for a forcing such that $\Vdash_{\mathbb{P}}$ "$\Vdash_{\mathring{\mathbb{Q}}} \mathcal{F}$ is big". Then $\Vdash_{\mathbb{P} * \mathring{\mathbb{Q}}} \mathcal{F}$ is big.*

Proof. Let K be $(\mathbf{V}, \mathbb{P} * \mathring{\mathbb{Q}})$-generic. Then there are G and H such that G is $(\mathbf{V}, \mathbb{P})$-generic, H is $(\mathbf{V}[G], \mathring{\mathbb{Q}}[G])$-generic and $\mathbf{V}[K] = \mathbf{V}[G][H]$. By hypothesis, in $\mathbf{V}[G]$ we have that $\mathcal{F}$ is big, $\mathring{\mathbb{Q}}[G]$ is a forcing, and $\Vdash_{\mathring{\mathbb{Q}}[G]} \mathcal{F}$ is big. Therefore in $\mathbf{V}[K] = \mathbf{V}[G][H]$, $\mathcal{F}$ is big. $\dashv$

The next lemma is a variation of Theorem 61 of Raghavan [35]. See also Raghavan [34].

Lemma 6.12. *Let $<_{\mathtt{wo}}$ be a well-ordering of $H(\aleph_0)$. Suppose $\mathcal{F} \subseteq H(\aleph_0)^\omega$ is well-closed w.r.t. $<_{\mathtt{wo}}$ and big. Let γ be a limit ordinal and let $\langle \mathbb{P}_\alpha; \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ be a CS iteration such that $\forall \alpha < \gamma \left[\Vdash_\alpha \mathring{\mathbb{Q}}_\alpha \text{ is proper}\right]$. Suppose that for all $\alpha < \gamma$, $\Vdash_\alpha \mathcal{F}$ is big. Then $\Vdash_\gamma \mathcal{F}$ is big.*

Proof. Let $\mathring{f} \in \mathbf{V}^{\mathbb{P}_\gamma}$ be such that $\Vdash_\gamma \mathring{f} \in H(\aleph_0)^\omega$ and let $p_0 \in \mathbb{P}_\gamma$. Fix a sufficiently large regular θ and a countable $M \prec H(\theta)$ with $\mathcal{F}, \langle \mathbb{P}_\alpha; \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle, \mathring{f}, p_0 \in M$. Since $\mathcal{F}$ satisfies (3) of Definition 6.8, we can find a slalom $S \in \mathcal{F}$ so that for all $f \in \mathcal{F} \cap M$, $\forall^\infty k \in \omega\, [f(k) \in S(k)]$. We will find a $q \in \mathbb{P}_\gamma$ such that $q \Vdash_\gamma p_0 \in \mathring{G}_\gamma$ and $q \Vdash_\gamma \exists^\infty n \in \omega \left[\mathring{f}(n) \in S(n)\right]$. Put $\gamma' = \sup(M \cap \gamma)$ and let $\langle \gamma_n : n \in \omega \rangle \subseteq M \cap \gamma$ be an increasing sequence that is cofinal in γ'. We build two sequences $\langle q_n : n \in \omega \rangle$ and $\langle \mathring{p}_n : n \in \omega \rangle$ satisfying the following conditions for all $n \in \omega$:

(1) $q_n \in \mathbb{P}_{\gamma_n}$, q_n is $(M, \mathbb{P}_{\gamma_n})$-generic, and $q_{n+1} \restriction \gamma_n = q_n$;
(2) $\mathring{p}_0 = \check{p}_0$, $\mathring{p}_n \in \mathbf{V}^{\mathbb{P}_{\gamma_n}}$, and $q_n \Vdash_{\gamma_n}$ "$\mathring{p}_n \in M \cap \mathbb{P}_\gamma \wedge \mathring{p}_n \restriction \gamma_n \in \mathring{G}_{\gamma_n}$";
(3) $q_{n+1} \Vdash_{\gamma_{n+1}} \mathring{p}_{n+1} \leq \mathring{p}_n$;
(4) $q_{n+1} \Vdash_{\gamma_{n+1}}$ "$\mathring{p}_{n+1} \Vdash_{\mathbb{P}_\gamma / \mathring{G}_{\gamma_{n+1}}} \exists k \geq n \left[\mathring{f}\left[\mathring{G}_{\gamma_{n+1}}\right](k) \in S(k)\right]$".

Assume for a moment that such sequences have been constructed. Then $\bigcup_{n \in \omega} q_n$ is a condition in $\mathbb{P}_{\gamma'}$. We extend $\bigcup_{n \in \omega} q_n$ to a condition q in $\mathbb{P}_\gamma$ by setting $q(\alpha) = \mathbb{1}_{\mathring{\mathbb{Q}}_\alpha}$ for all $\gamma' \leq \alpha < \gamma$. By standard arguments, $\forall n \in \omega \left[q \Vdash_\gamma \mathring{p}_n \in \mathring{G}_\gamma\right]$. In particular, $q \Vdash_\gamma p_0 \in \mathring{G}_\gamma$. We will check that $q \Vdash_\gamma \exists^\infty n \in \omega \left[\mathring{f}(n) \in S(n)\right]$.

Suppose not. Then $\exists r \leq q \exists n \in \omega \left[r \Vdash_\gamma \forall k \geq n \left[\mathring{f}(k) \notin S(k)\right]\right]$. Let G_γ be $(\mathbf{V}, \mathbb{P}_\gamma)$-generic with $r \in G_\gamma$. It is a standard fact that $G_{\gamma_{n+1}}$ is $(\mathbf{V}, \mathbb{P}_{\gamma_{n+1}})$-generic, that G_γ is $\left(\mathbf{V}\left[G_{\gamma_{n+1}}\right], \mathbb{P}_\gamma / G_{\gamma_{n+1}}\right)$-generic, and that in $\mathbf{V}[G_\gamma]$: $G_\gamma = G_{\gamma_{n+1}} * G_\gamma$. So in $\mathbf{V}\left[G_{\gamma_{n+1}}\right]$: $\mathring{p}_{n+1}\left[G_{\gamma_{n+1}}\right] \in \mathbb{P}_\gamma / G_{\gamma_{n+1}}$ and

$$\mathring{p}_{n+1}[G_\gamma] = \mathring{p}_{n+1}\left[G_{\gamma_{n+1}}\right] \Vdash_{\mathbb{P}_\gamma / G_{\gamma_{n+1}}} \exists k \geq n+1 \left[\mathring{f}\left[G_{\gamma_{n+1}}\right](k) \in S(k)\right].$$

Since $\mathring{p}_{n+1}[G_\gamma] \in G_\gamma$, in $\mathbf{V}[G_\gamma] = \mathbf{V}\left[G_{\gamma_{n+1}}\right][G_\gamma]$: there exists $k \geq n+1$ such that $\mathring{f}\left[G_{\gamma_{n+1}}\right][G_\gamma](k) \in S(k)$. Since $\mathring{f}\left[G_{\gamma_{n+1}}\right][G_\gamma] = \mathring{f}[G_\gamma]$, so $\mathring{f}[G_\gamma](k) \in S(k)$, and we have a contradiction.

To build $\langle q_n : n \in \omega \rangle$ and $\langle \mathring{p}_n : n \in \omega \rangle$, proceed by induction on n. Let $\mathring{p}_0 = \check{p}_0$, the canonical $\mathbb{P}_{\gamma_0}$-name for p_0. Note that $p_0 \restriction \gamma_0 \in M \cap \mathbb{P}_{\gamma_0}$ and $\mathbb{P}_{\gamma_0} \in M$. By the properness of $\mathbb{P}_{\gamma_0}$, there is $q_0 \in \mathbb{P}_{\gamma_0}$ which is $(M, \mathbb{P}_{\gamma_0})$-generic with $q_0 \leq p_0 \restriction \gamma_0$. Then $\mathring{p}_0$ and q_0 satisfy (1)–(4). Now suppose $\mathring{p}_n$ and q_n are given, for some $n \in \omega$. By the Properness Extension Lemma (see, for example, Lemma 2.8 of Abraham [1]), there is an $(M, \mathbb{P}_{\gamma_{n+1}})$-generic condition $q_{n+1} \in \mathbb{P}_{\gamma_{n+1}}$ such that $q_{n+1} \restriction \gamma_n = q_n$ and $q_{n+1} \Vdash_{\gamma_{n+1}} \mathring{p}_n \restriction \gamma_{n+1} \in \mathring{G}_{\gamma_{n+1}}$.

To find $\mathring{p}_{n+1}$ we proceed as follows. Fix a $(\mathbf{V}, \mathbb{P}_{\gamma_{n+1}})$-generic filter $G_{\gamma_{n+1}}$ with $q_{n+1} \in G_{\gamma_{n+1}}$. Recall that $M\left[G_{\gamma_{n+1}}\right] \prec H(\theta)\left[G_{\gamma_{n+1}}\right] = H^{\mathbf{V}\left[G_{\gamma_{n+1}}\right]}(\theta)$. Note that

$\mathbb{P}_\gamma / G_{\gamma_{n+1}} \in M\left[G_{\gamma_{n+1}}\right]$ and that in $\mathbf{V}\left[G_{\gamma_{n+1}}\right]$, $\Vdash_{\mathbb{P}_\gamma / G_{\gamma_{n+1}}} \mathring{f}\left[G_{\gamma_{n+1}}\right] \in H(\aleph_0)^\omega$. Since $q_{n+1} \in G_{\gamma_{n+1}}$, $\mathring{p}_n\left[G_{\gamma_{n+1}}\right] \restriction \gamma_{n+1} \in G_{\gamma_{n+1}}$. So in $\mathbf{V}\left[G_{\gamma_{n+1}}\right]$, $\mathring{p}_n\left[G_{\gamma_{n+1}}\right] \in M \cap \mathbb{P}_\gamma / G_{\gamma_{n+1}}$. By elementarity, find a sequence $\langle p^i : i \in \omega \rangle \in M\left[G_{\gamma_{n+1}}\right]$ and a function $f \in H(\aleph_0)^\omega \cap M\left[G_{\gamma_{n+1}}\right]$ satisfying the following for each $i \in \omega$:

(5) $p^i \in \mathbb{P}_\gamma / G_{\gamma_{n+1}}$;
(6) $p^{i+1} \leq p^i \leq \mathring{p}_n\left[G_{\gamma_{n+1}}\right]$;
(7) $p^i \Vdash_{\mathbb{P}_\gamma / G_{\gamma_{n+1}}} \mathring{f}\left[G_{\gamma_{n+1}}\right](i) = f(i)$.

By hypothesis, $\mathcal{F}$ is big in $\mathbf{V}\left[G_{\gamma_{n+1}}\right]$. So by elementarity and by the fact that $f \in M\left[G_{\gamma_{n+1}}\right]$, we can find $g \in \mathcal{F} \cap M\left[G_{\gamma_{n+1}}\right]$ so that $\exists^\infty k \in \omega\,[f(k) = g(k)]$. Since q_{n+1} is an $(M, \mathbb{P}_{\gamma_{n+1}})$-generic condition with $q_{n+1} \in G_{\gamma_{n+1}}$ and since $\mathcal{F} \in \mathbf{V}$, $\mathcal{F} \cap M = \mathcal{F} \cap M\left[G_{\gamma_{n+1}}\right]$. So $g \in \mathcal{F} \cap M$. By the choice of S, $\forall^\infty k \in \omega\,[g(k) \in S(k)]$. Hence we can fix $k \geq n+1$ such that $f(k) \in S(k)$. Note that $p^k \in \mathbb{P}_\gamma \cap M\left[G_{\gamma_{n+1}}\right]$, and that $\mathbb{P}_\gamma \cap M\left[G_{\gamma_{n+1}}\right] = \mathbb{P}_\gamma \cap M$ similarly to $\mathcal{F}$. Thus p^k has the following properties:

(8) $p^k \in M \cap \mathbb{P}_\gamma$ and $p^k \restriction \gamma_{n+1} \in G_{\gamma_{n+1}}$;
(9) $p^k \leq \mathring{p}_n\left[G_{\gamma_{n+1}}\right]$;
(10) $p^k \Vdash_{\mathbb{P}_\gamma / G_{\gamma_{n+1}}} \exists i \geq n+1 \left[\mathring{f}\left[G_{\gamma_{n+1}}\right](i) \in S(i)\right]$.

Since $G_{\gamma_{n+1}}$ was an arbitrary $(\mathbf{V}, \mathbb{P}_{\gamma_{n+1}})$-generic filter with $q_{n+1} \in G_{\gamma_{n+1}}$, we can use the maximal principle in $\mathbf{V}$ to find a $\mathring{p}_{n+1} \in \mathbf{V}^{\mathbb{P}_{\gamma_{n+1}}}$ so that in $\mathbf{V}$:

(11) $q_{n+1} \Vdash_{\mathbb{P}_{\gamma_{n+1}}}$ "$\mathring{p}_{n+1} \in M \cap \mathbb{P}_\gamma \wedge \mathring{p}_{n+1} \restriction \gamma_{n+1} \in \mathring{G}_{\gamma_{n+1}}$";
(12) $q_{n+1} \Vdash_{\mathbb{P}_{\gamma_{n+1}}} \mathring{p}_{n+1} \leq \mathring{p}_n$;
(13) $q_{n+1} \Vdash_{\mathbb{P}_{\gamma_{n+1}}}$ "$\mathring{p}_{n+1} \Vdash_{\mathbb{P}_\gamma / \mathring{G}_{\gamma_{n+1}}} \exists k \geq n+1 \left[\mathring{f}\left[\mathring{G}_{\gamma_{n+1}}\right](k) \in S(k)\right]$".

This concludes the inductive construction.

To complete the proof, fix an arbitrary $(\mathbf{V}, \mathbb{P}_\gamma)$-generic filter G_γ. In $\mathbf{V}[G_\gamma]$, by what has been proved above, for every $f \in H(\aleph_0)^\omega$, there exists a slalom $S \in \mathcal{F}$ so that $\exists^\infty k \in \omega\,[f(k) \in S(k)]$. Since $\mathbb{P}_\gamma$ is proper and since $\mathcal{F}$ is well-closed w.r.t. $<_{\texttt{wo}}$ in $\mathbf{V}$, $\mathcal{F}$ is still well-closed w.r.t. $<_{\texttt{wo}}$ in $\mathbf{V}[G_\gamma]$. So Lemma 6.7 applies in $\mathbf{V}[G_\gamma]$ and implies that $\forall f \in H(\aleph_0)^\omega \exists g \in \mathcal{F} \exists^\infty k \in \omega\,[f(k) = g(k)]$. ⊣

Corollary 6.13. *Let $<_{\texttt{wo}}$ be a well-ordering of $H(\aleph_0)$. Suppose $\mathcal{F} \subseteq H(\aleph_0)^\omega$ is well-closed w.r.t. $<_{\texttt{wo}}$ and is big. Let γ be any ordinal. Suppose $\langle \mathbb{P}_\alpha; \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ is a CS iteration such that $\forall \alpha < \gamma \left[\Vdash_\alpha \mathring{\mathbb{Q}}_\alpha \text{ is proper}\right]$. Suppose also that for each $\alpha < \gamma \Vdash_\alpha$ "$\Vdash_{\mathring{\mathbb{Q}}_\alpha} \mathcal{F}$ is big". Then $\Vdash_\gamma \mathcal{F}$ is big.*

Proof. The proof is by induction on γ. If $\gamma = 0$, then $\mathbb{P}_\gamma$ is the trivial forcing. By hypothesis, in $\mathbf{V}$, $\mathcal{F}$ is big, so there is nothing to do. Suppose $\gamma = \gamma' + 1$ and that the statement is true for γ'. So $\Vdash_{\gamma'} \mathcal{F}$ is big. Now $\mathbb{P}_{\gamma'+1}$ is forcing equivalent to $\mathbb{P}_{\gamma'} * \mathring{\mathbb{Q}}_{\gamma'}$. By hypothesis, $\Vdash_{\mathbb{P}_{\gamma'}}$ "$\Vdash_{\mathring{\mathbb{Q}}_{\gamma'}} \mathcal{F}$ is big". By Lemma 6.11, we have $\Vdash_{\mathbb{P}_{\gamma'} * \mathring{\mathbb{Q}}_{\gamma'}} \mathcal{F}$ is big. So $\Vdash_\gamma \mathcal{F}$ is big as required. Finally suppose that γ is a limit ordinal and that the statement is true for all $\beta < \gamma$. So $\forall \beta < \gamma\,[\Vdash_\beta \mathcal{F}$ is big$]$. By Lemma 6.12, $\Vdash_\gamma \mathcal{F}$ is big. This concludes the induction and the proof. ⊣

Lemma 6.14. *Suppose $<_{\texttt{wo}}$ is a well-ordering of $H(\aleph_0)$. If $\mathcal{F} \subseteq H(\aleph_0)^\omega$ is well-closed w.r.t. $<_{\texttt{wo}}$ and is big, then $\Vdash_{\mathbb{C}} \mathcal{F}$ is big.*

Proof. Let $\mathring{f}$ be a $\mathbb{C}$-name and suppose that $\Vdash_{\mathbb{C}} \mathring{f} : \omega \to H(\aleph_0)$. Enumerate $\mathbb{C}$ as $\{p_n : n \in \omega\}$ in a 1-1 way. Define a small slalom $S : \omega \to H(\aleph_0)$ as follows. Given $n \in \omega$, find $q_n \leq p_n$ and $x_n \in H(\aleph_0)$ so that $q_n \Vdash_{\mathbb{C}} \mathring{f}(n) = x_n$, and define $S(n) = \{x_n\}$. Note that for each $l \in \omega$, $X_l \in [\omega]^\omega$, where $X_l = \{n \in \omega : p_n \leq p_l\}$.

Since $\mathcal{F}$ is big, Lemma 6.5 applies and implies that there is a slalom $T : \omega \to H(\aleph_0)$ so that $T \in \mathcal{F}$ and $\forall l \in \omega \exists^\infty n \in X_l \, [S(n) \subseteq T(n)]$. Now we check that $\Vdash_{\mathbb{C}} \exists^\infty n \in \omega \left[\mathring{f}(n) \in T(n)\right]$. To see this, fix $p \in \mathbb{C}$ and $N \in \omega$. Then $p = p_l$ for some $l \in \omega$. Find $n \in X_l$ with $n > N$ and $S(n) \subseteq T(n)$. By the definition of S and X_l, $q_n \leq p_n \leq p_l$ and $q_n \Vdash_{\mathbb{C}} \mathring{f}(n) = x_n \in \{x_n\} = S(n) \subseteq T(n)$. This is as required.

Now to complete the proof, fix a $(\mathbf{V}, \mathbb{C})$-generic filter G. As $\mathbb{C}$ is proper, $\mathcal{F}$ is well-closed w.r.t. $<_{\mathtt{wo}}$ in $\mathbf{V}[G]$. By what has been proved in the previous paragraph, we have in $\mathbf{V}[G]$ that whenever $f : \omega \to H(\aleph_0)$, then there is a slalom $T \in \mathcal{F}$ with the property that $\exists^\infty n \in \omega \, [f(n) \in T(n)]$. As $\mathcal{F}$ is well-closed w.r.t. $<_{\mathtt{wo}}$ in $\mathbf{V}[G]$, Lemma 6.7 tells us that $\mathcal{F}$ is big in $\mathbf{V}[G]$. ⊣

Corollary 6.15. *Suppose $<_{\mathtt{wo}}$ is a well-ordering of $H(\aleph_0)$. If $\mathcal{F} \subseteq H(\aleph_0)^\omega$ is well-closed w.r.t. $<_{\mathtt{wo}}$ and is big, then $\Vdash_{\mathbb{C}_{\omega_1}} \mathcal{F}$ is big.*

Proof. Suppose G is $(\mathbf{V}, \mathbb{C}_{\omega_1})$-generic. Consider any $f \in H(\aleph_0)^\omega \cap \mathbf{V}[G]$. It is well-known that for some $(\mathbf{V}, \mathbb{C})$-generic filter H, $f \in \mathbf{V}[H]$. By Lemma 6.14, $\mathcal{F}$ is big in $\mathbf{V}[H]$. So there is $g \in \mathcal{F}$ so that $\exists^\infty n \in \omega \, [f(n) = g(n)]$. Therefore, $\mathcal{F}$ is big in $\mathbf{V}[G]$. ⊣

7. Shelah-Steprāns families and non-meager sets

In Subsections 7.1 and 7.2, we show that it is consistent to have $\mathrm{non}(\mathcal{M}) = \aleph_1$ and no Shelah-Steprāns or block Shelah-Steprāns a.d. families of size $\aleph_1$. This improves Theorem 5.6. This also shows that even though Shelah-Steprāns a.d. families are hard to destroy, they can still be diagonalized without increasing $\mathrm{non}(\mathcal{M})$ (see Corollary 7.17). The question of whether it is possible to do this for every maximal almost disjoint family, at least consistently, is open.

7.1. **The partial order.** In $\mathbf{V}$, let $\mathscr{A} \subseteq [\omega]^\omega$ be a fixed block Shelah-Steprāns a.d. family. Let $\mathbf{V}_{\omega_1}$ denote the extension of $\mathbf{V}$ by ω_1 Cohen reals (i.e. by $\mathbb{C}_{\omega_1}$). We assume that $\mathscr{A}$ remains block Shelah-Steprāns in $\mathbf{V}_{\omega_1}$. Actually, Shelah-Steprāns a.d. families remain Shelah-Steprāns after adding Cohen reals, but this does not seem to hold in general for block Shelah-Steprāns a.d. families. However, since we are mainly interested in the minimal cardinality of Shelah-Steprāns and block Shelah-Steprāns a.d. families, this is not relevant for our main result in the next section.

Definition 7.1. If F and G are non-empty sets of ordinals, we write $F < G$ as an abbreviation for $\forall x \in F \forall y \in G \, [x < y]$.

We use FIN to denote $[\omega]^{<\omega} \setminus \{\emptyset\}$.

Lemma 7.2. *Shelah-Steprāns a.d. families remain Shelah-Steprāns after adding Cohen reals.*

Proof. Let $\mathscr{B}$ be a Shelah-Steprāns a.d. family. It suffices to show that

$$\Vdash_{\mathbb{C}} \text{“}\mathscr{B} \text{ is Shelah-Steprāns''}.$$

To this end, let $\mathring{X}$ be a $\mathbb{C}$-name such that $\Vdash_{\mathbb{C}} \mathring{X} \subseteq \mathrm{FIN}$ and

$$\Vdash_{\mathbb{C}} \text{“}\forall B \in \mathcal{I}(\mathscr{B}) \exists s \in \mathring{X} \, [s \cap B = \emptyset]\text{''}.$$

Let $\langle p_n : n \in \omega \rangle$ enumerate $\mathbb{C}$. For each $n \in \omega$, let

$$X_n = \left\{ s \in \mathrm{FIN} : \exists q \leq p_n \left[q \Vdash_{\mathbb{C}} s \in \mathring{X} \right] \right\}.$$

By hypothesis, for any $n \in \omega$ and any $B \in \mathcal{I}(\mathscr{B})$, there exists $q \leq p_n$ and $s \in \mathrm{FIN}$ such that $s \cap B = \emptyset$ and $q \Vdash_{\mathbb{C}} s \in \mathring{X}$, whence $s \in X_n$. Since $\mathscr{B}$ is a Shelah-Steprāns a.d. family, there exists $B \in \mathcal{I}(\mathscr{B})$ with the property that

$\forall n \in \omega \left[[B]^{<\aleph_0} \cap X_n \text{ is infinite}\right]$. We check that $\Vdash_{\mathbb{C}}$ "$[B]^{<\aleph_0} \cap \mathring{X}$ is infinite". To this end, fix $p \in \mathbb{C}$ and some $k \in \omega$. Then $p = p_n$ for some $n \in \omega$, and there exists $s \in [B]^{<\aleph_0} \cap X_n$ such that $s \notin \mathcal{P}(k)$. By definition of X_n, $q \Vdash_{\mathbb{C}} s \in \mathring{X}$, for some $q \leq p_n = p$. Thus we have proved that for each $p \in \mathbb{C}$ and $k \in \omega$, there exist $s \in [B]^{<\aleph_0}$ and $q \leq p$ such that $s \notin \mathcal{P}(k)$ and $q \Vdash_{\mathbb{C}} s \in \mathring{X}$, which proves that $\Vdash_{\mathbb{C}}$ "$[B]^{<\aleph_0} \cap \mathring{X}$ is infinite". ⊣

We work in $\mathbf{V}_{\omega_1}$ for the remainder of this section unless the contrary is explicitly stated. As stated above, $\mathscr{A} \subseteq [\omega]^{\omega}$ is a fixed block Shelah-Steprāns a.d. family which is a member of $\mathbf{V}$ and remains block Shelah-Steprāns in $\mathbf{V}_{\omega_1}$.

Definition 7.3. Working in $\mathbf{V}_{\omega_1}$, define a forcing $\mathbb{P}(\mathscr{A})$ as follows. A pair $p = \langle s_p, c_p \rangle$ belongs to $\mathbb{P}(\mathscr{A})$ if and only if:

(1) $s_p \in [\omega]^{<\omega}$;
(2) $c_p : \omega \to \mathrm{FIN}$ such that:
 (2a) $\forall i \in \omega\, [c_p(i) < c_p(i+1)]$;
 (2b) $\forall x \in s_p \forall y \in c_p(0)\, [x < y]$;
(3) $\forall B \in \mathcal{I}(\mathscr{A}) \exists i \in \omega\, [B \cap c_p(i) = \emptyset]$.

For $q, p \in \mathbb{P}(\mathscr{A})$, $q \leq p$ if and only if:

(4) $s_q \supseteq s_p$;
(5) $\exists F_{q,p} \in [\omega]^{<\omega} \left[s_q \setminus s_p = \bigcup_{i \in F_{q,p}} c_p(i)\right]$;
(6) $\forall i \in \omega \exists G_{q,p,i} \in [\omega]^{<\omega} \left[c_q(i) = \bigcup_{j \in G_{q,p,i}} c_p(j)\right]$.

For $n \in \omega$ and $q, p \in \mathbb{P}(\mathscr{A})$, $q \leq_n p$ if and only if ($q \leq p$ and $c_q \restriction n = c_p \restriction n$). It is easy to see that $\leq$ and $\leq_n$ are transitive.

The following lemma lists some elementary consequences of the definitions which are easy to verify.

Lemma 7.4. *Let $p, q \in \mathbb{P}(\mathscr{A})$. The following hold:*

(1) $q \leq_0 p \iff q \leq p$;
(2) for all $n > 0$, if $q \leq_n p$, then $s_q = s_p$;
(3) if $q \leq p$, then for all $i < j < \omega$, $G_{q,p,i} < G_{q,p,j}$;
(4) $\forall n < m < \omega\, [q \leq_m p \implies q \leq_n p]$.

Lemma 7.5. *$\mathbb{P}(\mathscr{A})$ is non-empty.*

Proof. Working in $\mathbf{V}$, define a partial order $\mathbb{S}$ as follows. $s \in \mathbb{S}$ if and only if $s : n_s \to \mathrm{FIN}$, where $n_s \in \omega$ and $\forall i < i+1 < n_s\, [s(i) < s(i+1)]$. For $t, s \in \mathbb{S}$, $t \leq s$ if and only if $t \supseteq s$. It is easy to see that for each $i \in \omega$, $D_i = \{t \in \mathbb{S} : i < n_t\}$, and for each $B \in (\mathcal{I}(\mathscr{A}))^{\mathbf{V}}$, $E_B = \{t \in \mathbb{S} : \exists i < n_t\, [B \cap t(i) = \emptyset]\}$ are dense subsets of $\langle \mathbb{S}, \leq \rangle$ belonging to $\mathbf{V}$. In $\mathbf{V}_{\omega_1}$, there exists a $(\mathbf{V}, \mathbb{S})$-generic filter H. Setting $c = \bigcup H$, it is clear that $\langle \emptyset, c \rangle$ is a condition in $\mathbb{P}(\mathscr{A})$. ⊣

The relations $\leq_n$ do not define an Axiom A structure on $\mathbb{P}(\mathscr{A})$ because the limit of a fusion sequence will not, in general, satisfy clause (3) of Definition 7.3. However, the next lemma says that clause (3) of Definition 7.3 is the only obstruction. The proof is straightforward and left to the reader.

Lemma 7.6. *Suppose $\langle p_n : n \in \omega \rangle$ is a sequence of members of $\mathbb{P}(\mathscr{A})$ so that $\forall n \in \omega\, [p_{n+1} \leq_n p_n]$. Define $p = \langle s_p, c_p \rangle$ by setting $s_p = s_{p_1}$ and $c_p(n) = c_{p_{n+1}}(n)$, for all $n \in \omega$. Then clauses (1) and (2) of Definition 7.3 are satisfied. If clause (3) of Definition 7.3 is also satisfied, then $p \in \mathbb{P}(\mathscr{A})$ and $\forall n \in \omega\, [p \leq_n p_n]$.*

Definition 7.7. Let $p \in \mathbb{P}(\mathscr{A})$ and suppose $k \in \omega$ and $F \subseteq k$. Define $p(k, F) = \left\langle s_{p(k,F)}, c_{p(k,F)} \right\rangle$ by $s_{p(k,F)} = s_p \cup \left(\bigcup_{j \in F} c_p(j) \right)$ and $c_{p(k,F)}(i) = c_p(i + k)$, for all $i \in \omega$. It is easy to check that $p(k, F) \in \mathbb{P}(\mathscr{A})$ and that $p(k, F) \leq p$.

Definition 7.8. Let $p \in \mathbb{P}(\mathscr{A})$, $k \in \omega$, and $F \subseteq k$. Note that $p(k+1, F \cup \{k\}) \leq p$. Suppose that some $q \leq p(k+1, F \cup \{k\}) \leq p$ is given. Define $q(p, k, F) = \left\langle s_{q(p,k,F)}, c_{q(p,k,F)} \right\rangle$ as follows. Put $s_{q(p,k,F)} = s_p$. For $i < k$, define $c_{q(p,k,F)}(i) = c_p(i)$, define $c_{q(p,k,F)}(k) = \bigcup \{c_p(j) : j \in F_{q,p} \setminus k\}$, and for $i > k$, define $c_{q(p,k,F)}(i) = c_q(i - k - 1)$.

It is easy to check that $q(p, k, F) \in \mathbb{P}(\mathscr{A})$ and $q(p, k, F) \leq_k p$.

Lemma 7.9. *Let $p \in \mathbb{P}(\mathscr{A})$, $k \in \omega$, and $F \subseteq k$. Suppose $q \leq p(k+1, F \cup \{k\})$. If $r \leq q(p, k, F)$ and $i \in \omega$ are such that $k = \min\left(G_{r,q(p,k,F),i}\right)$ and $F_{r,q(p,k,F)} = F$, then $r(i+1, \{i\}) \leq q$.*

Proof. Note that $c_r(i) = \bigcup \left\{ c_{q(p,k,F)}(j) : j \in G_{r,q(p,k,F),i} \right\}$. Let

$$J = \left\{ j \in G_{r,q(p,k,F),i} : j > k \right\}.$$

Then

$$s_{r(i+1,\{i\})} = s_q \cup \bigcup \left\{ c_{q(p,k,F)}(j) : j \in J \right\} = s_q \cup \bigcup \{c_q(j - k - 1) : j \in J\}.$$

This confirms both (4) and (5) of Definition 7.3. For $l \in \omega$, we have $c_{r(i+1,\{i\})}(l) = c_r(l + i + 1) =$

$$\bigcup \left\{ c_{q(p,k,F)}(j) : j \in G_{r,q(p,k,F),l+i+1} \right\} = \bigcup \left\{ c_q(j - k - 1) : j \in G_{r,q(p,k,F),l+i+1} \right\}.$$

This confirms (6) of Definition 7.3. Therefore $r(i+1, \{i\}) \leq q$. $\dashv$

Lemma 7.10. *Let $p \in \mathbb{P}(\mathscr{A})$ and $k \in \omega$. Suppose $\mathring{x} \in \mathbf{V}_{\omega_1}^{\mathbb{P}(\mathscr{A})}$ and $\Vdash_{\mathbb{P}(\mathscr{A})} \mathring{x} \in \mathbf{V}_{\omega_1}$. Then there are $q \leq_k p$ and $X \in \mathbf{V}_{\omega_1}$ so that:*

(1) $|X| \leq 2^k$, $s_p = s_q$, and $c_q(k) \supseteq c_p(k)$;
(2) for any $q' \leq_{k+1} q$, $F \subseteq k$, $r \leq q'$, and $i \in \omega$, if $F_{r,q'} = F$ and $k = \min\left(G_{r,q',i}\right)$, then $r(i+1, \{i\}) \Vdash \mathring{x} \in X$.

Proof. Let $\left\langle F_l : l < 2^k \right\rangle$ enumerate $\mathcal{P}(k)$. We define by induction two sequences $\left\langle q_l : l \leq 2^k \right\rangle$ and $\left\langle x_l : l < 2^k \right\rangle$ satisfying the following: $q_0 = p$ and

$$\forall l' < l \left[q_l \leq_k q_{l'},\, s_{q_l} = s_{q_{l'}}, \text{ and } c_{q_l}(k) \supseteq c_{q_{l'}}(k) \right].$$

Define $q_0 = p$ and note that the induction hypothesis is vacuously satisfied. Now suppose $l < 2^k$ and that q_l satisfying the induction hypothesis is given. Then $q_l(k+1, F_l \cup \{k\}) \leq q_l$. Find $r_l \leq q_l(k+1, F_l \cup \{k\})$ and $x_l \in \mathbf{V}_{\omega_1}$ with $r_l \Vdash_{\mathbb{P}(\mathscr{A})} \mathring{x} = x_l$, and define $q_{l+1} = r_l(q_l, k, F_l)$. Note that $q_{l+1} \leq_k q_l$ and $s_{q_{l+1}} = s_{r_l(q_l,k,F_l)} = s_{q_l}$. Further, $c_{q_l}(k) \subseteq s_{q_l(k+1,F_l \cup \{k\})} \setminus s_{q_l} \subseteq s_{r_l} \setminus s_{q_l}$, and $k \in F_{r_l,q_l} \setminus k$. Thus $c_{q_l}(k) \subseteq c_{q_{l+1}}(k)$. Hence by the induction hypothesis,

$$\forall l' \leq l \left[q_{l+1} \leq_k q_l \leq_k q_{l'},\, s_{q_{l+1}} = s_{q_l} = s_{q_{l'}}, \text{ and } c_{q_{l+1}}(k) \supseteq c_{q_l}(k) \supseteq c_{q_{l'}}(k) \right].$$

This concludes the inductive construction.

Now define $q = q_{2^k}$ and $X = \{x_l : l < 2^k\}$. Then (1) is satisfied by construction. To verify (2), fix any $q' \leq_{k+1} q$, $F \subseteq k$, $r \leq q'$, and $i \in \omega$, and assume that $F_{r,q'} = F$ and that $k = \min\left(G_{r,q',i}\right)$. Then $F = F_l$, for some $l < 2^k$. Since $q \leq_k q_{l+1}$, $s_q = s_{q_{l+1}}$, and $c_q(k) \supseteq c_{q_{l+1}}(k)$, it follows that $F_{r,q_{l+1}} = F$ and that $\min\left(G_{r,q_{l+1},i}\right) = k$. By definition, $r_l \leq q_l(k+1, F \cup \{k\})$ and $q_{l+1} = r_l(q_l, k, F)$. Therefore, by Lemma 7.9, $r(i+1, \{i\}) \leq r_l$. Therefore, $r(i+1, \{i\}) \Vdash_{\mathbb{P}(\mathscr{A})} \mathring{x} = x_l \in X$. $\dashv$

Lemma 7.11. *Work in $\mathbf{V}_{\omega_1}$. Let θ be a sufficiently large regular cardinal. Suppose $M \prec H(\theta)$ is countable with M containing all relevant parameters. Let $f : \omega \to M$ be such that*

$$\forall k \in \omega \left[f(k) \in \mathbf{V}_{\omega_1}^{\mathbb{P}(\mathscr{A})} \wedge \Vdash_{\mathbb{P}(\mathscr{A})} f(k) \in \mathbf{V}_{\omega_1} \right].$$

For any $p \in \mathbb{P}(\mathscr{A}) \cap M$, there exist q and S satisfying the following:

(1) $q \leq p$;
(2) S *is a function,* $\operatorname{dom}(S) = \omega$, *and* $\forall k \in \omega \left[S(k) \subseteq M \wedge |S(k)| \leq 2^k \right]$;
(3) *for any* $k \in \omega$, *any* $F \subseteq k$, *any* $r \leq q$, *and any* $i \in \omega$, *if* $F_{r,q} = F$ *and* $\min(G_{r,q,i}) = k$, *then* $r(i+1, \{i\}) \Vdash_{\mathbb{P}(\mathscr{A})} f(k) \in S(k)$.

Proof. Let $\bar{M}$ denote the transitive collapse of M. Let $\pi : M \to \bar{M}$ be the collapsing map and $\pi^* : \bar{M} \to M$ be the inverse of π. Say that a is *an approximation* if:

(4) $a \in (\mathbb{P}(\mathscr{A}) \cap M)^{<\omega} \times \bar{M}^{<\omega}$;
(5) writing $a = \langle \sigma_a, \tau_a \rangle$, $\operatorname{dom}(\sigma_a) = \operatorname{dom}(\tau_a) + 1$;
(6) $\sigma_a(0) = p$ and $\forall n < n + 1 < \operatorname{dom}(\sigma_a) \left[\sigma_a(n+1) \leq_n \sigma_a(n) \right]$;
(7) for all $k < \operatorname{dom}(\tau_a)$, $|\tau_a(k)| \leq 2^k$;
(8) for any $k + 1 < \operatorname{dom}(\sigma_a)$, any $q' \leq_{k+1} \sigma_a(k+1)$, $F \subseteq k$, $r \leq q'$, and $i \in \omega$, if $F_{r,q'} = F$ and $k = \min(G_{r,q',i})$, then

$$r(i+1, \{i\}) \Vdash_{\mathbb{P}(\mathscr{A})} f(k) \in \{\pi^*(x) : x \in \tau_a(k)\}.$$

Let $\mathbb{A} = \{a : a \text{ is an approximation}\}$. Define a partial order on $\mathbb{A}$ by setting $\leq = \{\langle b, a \rangle \in \mathbb{A} \times \mathbb{A} : \sigma_b \supseteq \sigma_a \wedge \tau_b \supseteq \tau_a\}$. It is easy to see that $\langle \mathbb{A}, \leq \rangle \in H(\aleph_1)$.

For each $B \in \mathcal{I}^{\mathbf{V}}(\mathscr{A})$, define

$$D(B) = \left\{ b \in \mathbb{A} : \exists l \in \omega \left[l + 1 \in \operatorname{dom}(\sigma_b) \wedge c_{\sigma_b(l+1)}(l) \cap B = \emptyset \right] \right\}.$$

Claim 7.12. *$D(B)$ is dense in $\langle \mathbb{A}, \leq \rangle$.*

Proof. Let $a \in \mathbb{A}$ be given and put $k = \operatorname{dom}(\tau_a)$. Let $p' = \sigma_a(k)$ and $\mathring{x} = f(k)$. Note that $\mathring{x} \in M \cap \mathbf{V}_{\omega_1}^{\mathbb{P}(\mathscr{A})}$ and that $\Vdash_{\mathbb{P}(\mathscr{A})} \mathring{x} \in \mathbf{V}_{\omega_1}$. Note also that $p' \in M$. Let $Z = \{z \in \omega : z \geq k \wedge c_{p'}(z) \cap B = \emptyset\}$. By the definition of $\mathbb{P}(\mathscr{A})$, $Z \in [\omega]^{\omega}$. Let $\langle z_l : l \in \omega \rangle$ be the strictly increasing enumeration of Z. Define $p'' = \langle s_{p''}, c_{p''} \rangle$ by setting $s_{p''} = s_{p'}$, $c_{p''}(i) = c_{p'}(i)$, for all $i < k$, and $c_{p''}(k+l) = c_{p'}(z_l)$, for all $l \in \omega$. Then it is clear that $p'' \in \mathbb{P}(\mathscr{A})$ and that $p'' \leq_k p'$. Applying Lemma 7.10 with p'', k, and $\mathring{x}$, find $q^* \leq_k p'' \leq_k p'$ and $X^* \in \mathbf{V}_{\omega_1}$ satisfying (1) and (2) of Lemma 7.10 with respect to p'', k, and $\mathring{x}$. Notice that $c_{q^*}(k) \cap B = \emptyset$. p'' may not be in M, and so q^* and X^* may not be in M either. However, $p', \mathring{x}, c_{q^*}(k) \in M$. Therefore by elementarity and by the fact that M contains all the relevant parameters, there exist $q, X \in M$ satisfying the following properties:

(9) $q \leq_k p'$ and $|X| \leq 2^k$;
(10) for any $q' \leq_{k+1} q$, $F \subseteq k$, $r \leq q'$, and $i \in \omega$, if $F_{r,q'} = F$ and $k = \min(G_{r,q',i})$, then $r(i+1, \{i\}) \Vdash_{\mathbb{P}(\mathscr{A})} \mathring{x} \in X$;
(11) $c_q(k) = c_{q^*}(k)$.

Define $\sigma_b = \sigma_a \cup \{\langle k+1, q \rangle\}$. Next, $X \subseteq M$, and so $\{\pi(x) : x \in X\} \subseteq \bar{M}$ and $|\{\pi(x) : x \in X\}| = |X| \leq 2^k$. Since $\bar{M}$ is a transitive model of a sufficiently large fragment of ZFC $-$ P and since $\{\pi(x) : x \in X\}$ is a finite subset of $\bar{M}$, $\{\pi(x) : x \in X\} \in \bar{M}$. Define $\tau_b = \tau_a \cup \{\langle k, \{\pi(x) : x \in X\} \rangle\}$. Observe that $\{\pi^*(y) : y \in \tau_b(k)\} = X$. Using these observations and (10), it is easy to verify that $b = \langle \sigma_b, \tau_b \rangle$ satisfies (4)–(8). Therefore $b \in \mathbb{A}$ and $b \leq a$. Finally, $c_{\sigma_b(k+1)}(k) \cap B = c_{q^*}(k) \cap B = \emptyset$ because of (11). Therefore $l = k$ witnesses that $b \in D(B)$, establishing the density of $D(B)$. $\dashv$

Now note that $\mathbb{A}$ is non-empty because $\langle \{\langle 0, p\rangle\}, \emptyset\rangle \in \mathbb{A}$. As we have observed earlier, $\langle \mathbb{A}, \leq\rangle \in H(\aleph_1)$. Hence there exists $\delta < \omega_1$ such that $\langle \mathbb{A}, \leq\rangle \in \mathbf{V}_\delta$, where $\mathbf{V}_\delta$ is the extension of $\mathbf{V}$ by the first δ Cohen reals. Further, for $B \in \mathcal{I}^{\mathbf{V}}(\mathscr{A})$, $D(B) \in \mathbf{V}_\delta$ because $D(B)$ has an absolute definition in terms of the parameters $\mathbb{A}$ and B. Since $\langle \mathbb{A}, \leq\rangle$ is a countable forcing notion, there exists $H \in \mathbf{V}_{\omega_1}$ which is $(\mathbf{V}_\delta, \mathbb{A})$-generic. Define $P = \bigcup\{\sigma_a : a \in H\}$ and $T = \bigcup\{\tau_a : a \in H\}$. Then P and T are functions and $\mathrm{dom}(P) = \sup\{\mathrm{dom}(\sigma_a) : a \in H\}$.

Claim 7.13. $\mathrm{dom}(P) = \omega$.

Proof. Suppose not. Then $\mathrm{dom}(P) < \omega$. Let $B = \bigcup\left\{c_{P(l+1)}(l) : l+1 \in \mathrm{dom}(P)\right\}$. Since $\mathrm{dom}(P)$ is finite, B is a finite subset of ω, whence $B \in \mathcal{I}^{\mathbf{V}}(\mathscr{A})$. Consider $a \in H \cap D(B)$. By definition of $D(B)$, there exists $l+1 \in \mathrm{dom}(\sigma_a)$ such that $c_{\sigma_a(l+1)}(l) \cap B = \emptyset$. It follows that $l+1 \in \mathrm{dom}(P)$ and $B \supseteq c_{P(l+1)}(l) = c_{\sigma_a(l+1)}(l)$, whence $c_{\sigma_a(l+1)}(l) = \emptyset$. But this is impossible by Clause (2) of Definition 7.3. $\dashv$

For $n \in \omega$, define $p_n = P(n)$. Then $p_0 = p$, $\forall n \in \omega\, [p_n \in \mathbb{P}(\mathscr{A})]$, and $\forall n \in \omega\, [p_{n+1} \leq_n p_n]$. Define $q' = \langle s_{q'}, c_{q'}\rangle$ by setting $s_{q'} = s_{p_1}$ and $c_{q'}(n) = c_{p_{n+1}}(n)$, for all $n \in \omega$. Now if $B \in \mathcal{I}^{\mathbf{V}}(\mathscr{A})$, then for some $b \in H$ and for some $l \in \omega$, $c_{p_{l+1}}(l) \cap B = \emptyset$, whence $c_{q'}(l) \cap B = \emptyset$. Thus by Lemma 7.6, $q' \in \mathbb{P}(\mathscr{A})$ and $\forall n \in \omega\, [q' \leq_n p_n]$. In particular, $q' \leq_0 p_0 = p$, which is to say that $q' \leq p$. Next for each $k \in \omega$, $T(k) \in \bar{M}$ and $|T(k)| \leq 2^k$. $T(k) \subseteq \bar{M}$ because $\bar{M}$ is transitive. Defining $S(k) = \{\pi^*(y) : y \in T(k)\} \subseteq M$, $|S(k)| = |\{\pi^*(y) : y \in T(k)\}| = |T(k)| \leq 2^k$. Thus (1) and (2) are satisfied by q' and S. To see that (3) is satisfied, fix $k \in \omega$, $F \subseteq k$, $r \leq q'$, and $i \in \omega$. Assume that $F_{r,q'} = F$ and $\min(G_{r,q',i}) = k$. Note that $q' \leq_{k+1} p_{k+1}$ and that $p_{k+1} = \sigma_a(k+1)$, for some $a \in H$. Hence by (8),

$$r(i+1, \{i\}) \Vdash_{\mathbb{P}(\mathscr{A})} f(k) \in \{\pi^*(x) : x \in \tau_a(k)\} = \{\pi^*(x) : x \in T(k)\} = S(k).$$

This concludes the proof. $\dashv$

We remark that even though the filter H and the function T belong to an intermediate extension of the form $\mathbf{V}_\delta$, for some $\delta < \omega_1$, this is not the case for S. This is because the functions π and π^* are not in any intermediate $\mathbf{V}_\delta$ with $\delta < \omega_1$.

Corollary 7.14. *$\mathbb{P}(\mathscr{A})$ is proper.*

Proof. Working in $\mathbf{V}_{\omega_1}$, fix a countable $M \prec H(\theta)$, where θ is a sufficiently large regular cardinal and M contains the relevant parameters. Let $\langle \mathring{\alpha}_n : n \in \omega\rangle$ be an enumeration of all $\mathring{\alpha} \in M$ such that $\mathring{\alpha}$ is a $\mathbb{P}(\mathscr{A})$-name and $\Vdash_{\mathbb{P}(\mathscr{A})}$ "$\mathring{\alpha}$ is an ordinal". Define a function $f : \omega \to M$ as follows. Suppose $k \in \omega$. Consider a $(\mathbf{V}_{\omega_1}, \mathbb{P}(\mathscr{A}))$-generic filter G. In $\mathbf{V}_{\omega_1}[G]$, $\{\mathring{\alpha}_0[G], \dots, \mathring{\alpha}_k[G]\}$ is a finite set of ordinals and so $\{\mathring{\alpha}_0[G], \dots, \mathring{\alpha}_k[G]\} \in \mathbf{V}_{\omega_1}$. Applying the maximal principal in $\mathbf{V}_{\omega_1}$, there is a $\mathbb{P}(\mathscr{A})$-name $\mathring{x}$ such that $\Vdash_{\mathbb{P}(\mathscr{A})} \mathring{x} \in \mathbf{V}_{\omega_1}$ and

$$\Vdash_{\mathbb{P}(\mathscr{A})} \text{"}\forall y\, [y \in \mathring{x} \leftrightarrow (y = \mathring{\alpha}_0 \vee \dots \vee y = \mathring{\alpha}_k)]\text{"}.$$

Since $\mathring{\alpha}_0, \dots, \mathring{\alpha}_k \in M$, we may choose such an $\mathring{x} \in M$. Define $f(k) = \mathring{x} \in M$. Notice that $\Vdash_{\mathbb{P}(\mathscr{A})}$ "$f(k)$ is finite" and that $\Vdash_{\mathbb{P}(\mathscr{A})} \mathring{\alpha}_l \in f(k)$, for every $l \leq k$.

Unfix k from the previous paragraph. Fix any $p \in \mathbb{P}(\mathscr{A}) \cap M$. We must find $q \leq p$ which is $(M, \mathbb{P}(\mathscr{A}))$-generic. Applying Lemma 7.11, find q and S satisfying (1)–(3) of Lemma 7.11. We argue that q is $(M, \mathbb{P}(\mathscr{A}))$-generic. To this end, it suffices to show that for any $\mathring{\alpha} \in M$, if $\Vdash_{\mathbb{P}(\mathscr{A})}$ "$\mathring{\alpha}$ is an ordinal", then $q \Vdash_{\mathbb{P}(\mathscr{A})} \mathring{\alpha} \in M$. Let a relevant $\mathring{\alpha}$ be given. Then $\mathring{\alpha} = \mathring{\alpha}_l$, for some $l < \omega$. Suppose $r \leq q$. Let $F = F_{r,q}$. Since $\langle \min(G_{r,q,i}) : i \in \omega\rangle$ is a strictly increasing sequence, it is possible to find $k, i \in \omega$ such that $k = \min(G_{r,q,i})$, $F \subseteq k$, and $k \geq l$. By (3) of Lemma 7.11, $r(i+1, \{i\}) \Vdash_{\mathbb{P}(\mathscr{A})} f(k) \in S(k) \subseteq M$. Find $r' \leq r(i+1, \{i\}) \leq r$ and $X \in M$ with $r' \Vdash_{\mathbb{P}(\mathscr{A})} f(k) = X$. Since we know $\Vdash_{\mathbb{P}(\mathscr{A})}$ "$f(k)$ is finite", it follows that X

is finite, and since $X \in M$, $X \subseteq M$. Thus $r' \Vdash_{\mathbb{P}(\mathscr{A})} \mathring{\alpha}_l \in f(k) = X \subseteq M$, whence $r' \Vdash_{\mathbb{P}(\mathscr{A})} \mathring{\alpha}_l \in M$. Thus we have proved that $\forall r \leq q \exists r' \leq r \left[r' \Vdash_{\mathbb{P}(\mathscr{A})} \mathring{\alpha}_l \in M\right]$, whence $q \Vdash_{\mathbb{P}(\mathscr{A})} \mathring{\alpha}_l \in M$. This proves that $\mathbb{P}(\mathscr{A})$ is proper. $\dashv$

Lemma 7.15. *In* $\mathbf{V}$, *suppose* $<_{\mathtt{wo}}$ *is a well-ordering of* $H(\aleph_0)$, *and that* $\mathcal{F} \subseteq H(\aleph_0)^\omega$ *is well-closed w.r.t.* $<_{\mathtt{wo}}$ *and is big. Then in* $\mathbf{V}_{\omega_1}$, $\Vdash_{\mathbb{P}(\mathscr{A})} \mathcal{F}$ *is big.*

Proof. Work in $\mathbf{V}_{\omega_1}$. $\mathcal{F}$ is well-closed w.r.t. $<_{\mathtt{wo}}$ because $\mathbb{C}_{\omega_1}$ is proper and $\mathcal{F}$ is big by Lemma 6.15. Fix a $\mathbb{P}(\mathscr{A})$ name $\mathring{g}$ and assume that $\Vdash_{\mathbb{P}(\mathscr{A})} \mathring{g} : \omega \to H(\aleph_0)$. Let $p \in \mathbb{P}(\mathscr{A})$ be fixed. Let θ be a sufficiently large regular cardinal. Suppose $M \prec H(\theta)$ is countable with M containing all the relevant parameters. In particular, $\mathring{g}, p \in M$. Define $f : \omega \to M$ as follows. For each $k \in \omega$, find a $\mathbb{P}(\mathscr{A})$-name $\mathring{x} \in M$ such that $\Vdash_{\mathbb{P}(\mathscr{A})} \mathring{x} \in H(\aleph_0)$ and $\Vdash_{\mathbb{P}(\mathscr{A})} \mathring{g}(k) = \mathring{x}$, and define $f(k) = \mathring{x}$. Applying Lemma 7.11, find q and S satisfying (1)–(3) of Lemma 7.11. Note that for each $k \in \omega$, $|S(k) \cap H(\aleph_0)| \leq |S(k)| \leq 2^k$, and so $S(k) \cap H(\aleph_0)$ is a finite subset of $H(\aleph_0)$, which implies that $S(k) \cap H(\aleph_0) \in H(\aleph_0)$. Hence we may define a small slalom $S^* : \omega \to H(\aleph_0)$ by $S^*(k) = S(k) \cap H(\aleph_0)$, for all $k \in \omega$.

Next, define sequences $\langle \mathscr{A}_l : l \in \omega \rangle \subseteq [\mathscr{A}]^{<\aleph_0}$ and $\langle Y_l : l \in \omega \rangle \subseteq [\omega]^\omega$ as follows. Fix $l \in \omega$ and suppose that $\langle \mathscr{A}_{l'} : l' < l \rangle \subseteq [\mathscr{A}]^{<\aleph_0}$ and $\langle Y_{l'} : l' < l \rangle \subseteq [\omega]^\omega$ are already given. By the definition of $\mathbb{P}(\mathscr{A})$, the family

$$\mathcal{E} = \left\{ s \in \operatorname{ran}(c_q) : s \cap \left(\bigcup \left(\bigcup\nolimits_{l' < l} \mathscr{A}_{l'} \right) \right) = \emptyset \right\}$$

has the property that $\forall B \in \mathcal{I}(\mathscr{A}) \exists s \in \mathcal{E}\, [s \cap B = \emptyset]$. Since $\mathscr{A}$ is assumed to be block Shelah-Steprāns in $\mathbf{V}_{\omega_1}$, there exists $B \in \mathcal{I}(\mathscr{A})$ with the property that $\{y \in \omega : c_q(y) \in \mathcal{E} \wedge c_q(y) \subseteq B\}$ is infinite. Clearly for every $F \in [\omega]^{<\omega}$, the set $\{y \in \omega : c_q(y) \in \mathcal{E} \wedge c_q(y) \subseteq B \setminus F\}$ is still infinite. Therefore, there exists $\mathscr{A}^* \in [\mathscr{A}]^{<\aleph_0}$ such that $\{y \in \omega : c_q(y) \in \mathcal{E} \wedge c_q(y) \subseteq \bigcup \mathscr{A}^*\}$ is infinite. Let $\mathscr{A}_l$ be such an $\mathscr{A}^*$ of minimal cardinality. Define $Y_l = \{y \in \omega : c_q(y) \in \mathcal{E} \wedge c_q(y) \subseteq \bigcup \mathscr{A}_l\} \in [\omega]^\omega$. Suppose that $A \in \mathscr{A}_{l'} \cap \mathscr{A}_l$, for some $l' < l$. Then for each $y \in Y_l$, $c_q(y) \cap A = \emptyset$ and $c_q(y) \subseteq \bigcup \mathscr{A}_l$, whence $c_q(y) \subseteq \bigcup (\mathscr{A}_l \setminus \{A\})$, contradicting the minimality of $\mathscr{A}_l$. Therefore $\forall l' < l\, [\mathscr{A}_{l'} \cap \mathscr{A}_l = \emptyset]$. This concludes the definition of $\langle \mathscr{A}_l : l \in \omega \rangle$ and $\langle Y_l : l \in \omega \rangle$.

Since $\mathcal{F}$ is big and is well-closed w.r.t. $<_{\mathtt{wo}}$ in $\mathbf{V}_{\omega_1}$, Lemma 6.5 applies and implies that there exists $T \in \mathcal{F}$ such that $T : \omega \to H(\aleph_0)$ is a slalom and $\forall l \in \omega \exists^\infty y \in Y_l\, [S^*(y) \subseteq T(y)]$. Define $Z_l = \{y \in Y_l : S^*(y) \subseteq T(y)\} \in [\omega]^\omega$, for all $l \in \omega$. Let $Z = \bigcup_{l \in \omega} Z_l$ and let $\langle z_j : j < \omega \rangle$ be the strictly increasing enumeration of Z. Define $q^* = \langle s_{q^*}, c_{q^*} \rangle$ by $s_{q^*} = s_q$ and $c_{q^*}(j) = c_q(z_j)$, for all $j \in \omega$. Consider any $B \in \mathcal{I}^{\mathbf{V}}(\mathscr{A})$. Since $\forall l' < l < \omega\, [\mathscr{A}_{l'} \cap \mathscr{A}_l = \emptyset]$, there exists $l, m \in \omega$ such that $B \cap (\bigcup \mathscr{A}_l) \subseteq m$. As Z_l is infinite, we can find $y \in Z_l$ with $\min(c_q(y)) \geq m$. As $y \in Y_l$, $c_q(y) \subseteq \bigcup \mathscr{A}_l$, whence $B \cap c_q(y) = \emptyset$. Now $y = z_j$ for some $j < \omega$ because $y \in Z$. Therefore, $c_{q^*}(j) \cap B = c_q(z_j) \cap B = c_q(y) \cap B = \emptyset$. This shows that $q^* \in \mathbb{P}(\mathscr{A})$ and that $q^* \leq q$.

We will now argue that $q^* \Vdash_{\mathbb{P}(\mathscr{A})} \exists^\infty k \in \omega\, [\mathring{g}(k) \in T(k)]$. To this end fix $r \leq q^*$ and some $l \in \omega$. As $r \leq q$, $\langle \min(G_{r,q,i}) : i < \omega \rangle$ is a strictly increasing sequence. Let $F_{r,q} = F$. Find $k, i \in \omega$ such that $k = \min(G_{r,q,i})$, $F \subseteq k$, and $k \geq l$. It follows from the fact that $r \leq q^* \leq q$ and from the definition of q^* that $k \in Z$. In particular, $S^*(k) \subseteq T(k)$. By (3) of Lemma 7.11,

$$r(i+1, \{i\}) \Vdash_{\mathbb{P}(\mathscr{A})} \mathring{g}(k) = f(k) \in S(k) \cap H(\aleph_0) = S^*(k) \subseteq T(k),$$

whence $r(i+1, \{i\}) \Vdash_{\mathbb{P}(\mathscr{A})} \mathring{g}(k) \in T(k)$. Since $r(i+1, \{i\}) \leq r$, we have proved that for each $r \leq q^*$ and $l \in \omega$, there exists $k \geq l$ and $r' \leq r$ with $r' \Vdash_{\mathbb{P}(\mathscr{A})} \mathring{g}(k) \in T(k)$. This proves that $q^* \Vdash_{\mathbb{P}(\mathscr{A})} \exists^\infty k \in \omega\, [\mathring{g}(k) \in T(k)]$. Since $q^* \leq q \leq p$, we have now proved that for any $\mathring{g} \in \mathbf{V}_{\omega_1}^{\mathbb{P}(\mathscr{A})}$ and any $p \in \mathbb{P}(\mathscr{A})$, if $\Vdash_{\mathbb{P}(\mathscr{A})} \mathring{g} : \omega \to H(\aleph_0)$, then there exist $q^* \leq p$ and $T \in \mathcal{F}$ such that $T : \omega \to H(\aleph_0)$ is a slalom and $q^* \Vdash_{\mathbb{P}(\mathscr{A})} \exists^\infty k \in \omega \, [\mathring{g}(k) \in T(k)]$.

To complete the proof, let G be a $(\mathbf{V}_{\omega_1}, \mathbb{P}(\mathscr{A}))$-generic filter. As $\mathbb{P}(\mathscr{A})$ is proper, $\mathcal{F}$ is well-closed w.r.t. $<_{\mathtt{wo}}$ in $\mathbf{V}_{\omega_1}[G]$. By what has been proved above, we have in $\mathbf{V}_{\omega_1}[G]$ that for every $g : \omega \to H(\aleph_0)$, there exists $T \in \mathcal{F}$ such that $T : \omega \to H(\aleph_0)$ is a slalom and $\exists^\infty k \in \omega \, [g(k) \in T(k)]$. As $\mathcal{F}$ is well-closed w.r.t. $<_{\mathtt{wo}}$ in $\mathbf{V}_{\omega_1}[G]$, Lemma 6.7 tells us that $\mathcal{F}$ is big in $\mathbf{V}_{\omega_1}[G]$. ⊣

Note the similarity in the proofs of Lemmas 7.15 and 6.14. Also, the proof of Lemma 7.15 is the only place where we use the assumption that $\mathscr{A}$ is block Shelah-Steprāns.

Lemma 7.16. *$\mathbb{P}(\mathscr{A})$ diagonalizes $\mathscr{A}$.*

Proof. Let $\mathring{A}$ be a $\mathbb{P}(\mathscr{A})$-name such that $\Vdash_{\mathbb{P}(\mathscr{A})} \mathring{A} = \bigcup \left\{ s_p : p \in \mathring{G} \right\}$, where $\mathring{G}$ is the canonical $\mathbb{P}(\mathscr{A})$-name for a generic filter over $\mathbb{P}(\mathscr{A})$. Suppose $p \in \mathbb{P}(\mathscr{A})$ and $B \in \mathscr{A}$ are given. Then $Z = \{z \in \omega : c_p(z) \cap B = \emptyset\} \in [\omega]^\omega$. Let $\langle z_j : j < \omega \rangle$ be the strictly increasing enumeration of Z. Define $q = \langle s_q, c_q \rangle$ by setting $s_q = s_p$ and $c_q(j) = c_p(z_j)$, for all $j < \omega$. Then $q \leq p$ and $q \Vdash_{\mathbb{P}(\mathscr{A})} \left| \mathring{A} \cap B \right| < \aleph_0$. ⊣

The following corollary is worth stating even though it is not directly used in the proof of our main result in the next section.

Corollary 7.17. *Any Shelah-Steprāns a.d. family in* $\mathbf{V}$ *can be diagonalized without increasing* $\mathrm{non}(\mathcal{M})$*. Any block Shelah-Steprāns a.d. family in* $\mathbf{V}$ *which remains block Shelah-Steprāns after adding Cohen reals can be diagonalized without increasing* $\mathrm{non}(\mathcal{M})$*.*

Proof. If $\mathscr{A}$ is a Shelah-Steprāns a.d. family, then $\mathscr{A}$ remains Shelah-Steprāns, and hence block Shelah-Steprāns, after adding Cohen reals (see Lemma 7.2). Hence $\mathbb{C}_{\omega_1} * \mathring{\mathbb{P}}(\mathscr{A})$ will diagonalize $\mathscr{A}$ and not increase $\mathrm{non}(\mathcal{M})$. ⊣

7.2. A model where $\mathrm{non}(\mathcal{M}) = \aleph_1$ and there are no block Shelah-Steprāns a.d. families of size $\aleph_1$.

Theorem 7.18. *There is a model in which* $\mathrm{non}(\mathcal{M}) = \aleph_1$ *and there are no Shelah-Steprāns or block Shelah-Steprāns a.d. families of size* $\aleph_1$*.*

Proof. Shelah-Steprāns a.d. families are block Shelah-Steprāns. So it is enough to produce a model where $\mathrm{non}(\mathcal{M}) = \aleph_1$ and there are no block Shelah-Steprāns a.d. families of size $\aleph_1$. Let $\mathbf{V}$ be a universe satisfying GCH. In $\mathbf{V}$, let $<_{\mathtt{wo}}$ be a well-ordering of $H(\aleph_0)$ and let $\mathcal{F} = \mathbf{V} \cap H(\aleph_0)^\omega$. Then $\mathcal{F}$ is well-closed w.r.t. $<_{\mathtt{wo}}$ in $\mathbf{V}$. $\mathcal{F}$ is also big in $\mathbf{V}$. Build a CS iteration $\left\langle \mathbb{P}_\alpha; \mathring{\mathbb{Q}}_\alpha : \alpha \leq \omega_2 \right\rangle$ as follows. Using GCH in $\mathbf{V}$, fix a bookkeeping device which has the property that any $\mathbb{P}_{\omega_2}$-name for a set of reals of size $\aleph_1$ will be enumerated cofinally often. At a stage $\alpha < \omega_2$, suppose $\mathbb{P}_\alpha$ is given. Assume $\mathbb{P}_\alpha$ is proper and $\Vdash_\alpha \mathcal{F}$ is big. Suppose the bookkeeping device hands us a $\mathbb{P}_\alpha$-name $\mathring{\mathscr{A}}$ such that

$$\Vdash_\alpha \text{“}\mathring{\mathscr{A}} \subseteq [\omega]^\omega \text{ is an infinite a.d. family''}.$$

Let G_α be a $(\mathbf{V}, \mathbb{P}_\alpha)$-generic filter. In $\mathbf{V}[G_\alpha]$, either

$$\Vdash_{\mathbb{C}_{\omega_1}} \text{“}\mathring{\mathscr{A}}[G_\alpha] \text{ is block Shelah-Steprāns''}$$

or $\Vdash_{\mathbb{C}_{\omega_1}}$ “$\mathring{\mathscr{A}}[G_\alpha]$ is not block Shelah-Steprāns$''$ because $\mathbb{C}_{\omega_1}$ is almost homogeneous. If the first alternative happens, then let $\mathbb{Q}$ be $\mathbb{C}_{\omega_1} * \mathring{\mathbb{P}}(\mathring{\mathscr{A}}[G_\alpha])$. If the second alternative happens, then let $\mathbb{Q}$ be $\mathbb{C}_{\omega_1}$. Back in $\mathbf{V}$, let $\mathring{\mathbb{Q}}_\alpha$ be a full

$\mathbb{P}_\alpha$-name for $\mathbb{Q}$. If the bookkeeping device does not hand us a $\mathbb{P}_\alpha$-name of the form $\mathring{\mathscr{A}}$, then we let $\mathring{\mathbb{Q}}_\alpha$ be a full $\mathbb{P}_\alpha$-name for the trivial forcing. Observe that $\Vdash_\alpha$ "$\Vdash_{\mathring{\mathbb{Q}}_\alpha} \mathring{\mathscr{A}}$ is not block Shelah-Steprāns" (if an appropriate $\mathring{\mathscr{A}}$ is given) and that $\Vdash_\alpha$ "$\Vdash_{\mathring{\mathbb{Q}}_\alpha} \mathcal{F}$ is big". This concludes the construction.

Let G_{ω_2} be a $(\mathbf{V}, \mathbb{P}_{\omega_2})$-generic filter. In $\mathbf{V}[G_{\omega_2}]$, by Corollary 6.13, $\mathcal{F}$ is big. Therefore by Lemma 6.2, $2^\omega \cap \mathbf{V}$ is non-meager in $\mathbf{V}[G_{\omega_2}]$. Therefore $\mathrm{non}(\mathcal{M}) = \aleph_1$ in $\mathbf{V}[G_{\omega_2}]$.

Next, suppose for a contradiction that in $\mathbf{V}[G_{\omega_2}]$, there exists $\mathscr{A} \subseteq [\omega]^\omega$ which is a block Shelah-Steprāns a.d. family with $|\mathscr{A}| = \aleph_1$. For $\gamma \leq \omega_2$, G_γ denotes the restriction of G_{ω_2} to $\mathbb{P}_\gamma$. There exists $\xi < \omega_2$ such that $\mathscr{A} \in \mathbf{V}[G_\xi]$. Since the property of being a block Shelah-Steprāns a.d. family is downwards absolute, $\mathscr{A}$ is a block Shelah-Steprāns a.d. family in $\mathbf{V}[G_\gamma]$, for all $\xi \leq \gamma \leq \omega_2$. The bookkeeping device ensured that for some $\xi \leq \alpha < \omega_2$, a $\mathbb{P}_\alpha$-name $\mathring{\mathscr{A}}$ such that $\mathring{\mathscr{A}}[G_\alpha] = \mathscr{A}$ was considered at stage α. By the choice of $\mathring{\mathbb{Q}}_\alpha$, $\Vdash_{\alpha+1}$ "$\mathring{\mathscr{A}}$ is not block Shelah-Steprāns". This is a contradiction because $\mathscr{A} = \mathring{\mathscr{A}}[G_\alpha] = \mathring{\mathscr{A}}[G_{\alpha+1}]$ is block Shelah-Steprāns in $\mathbf{V}[G_{\alpha+1}]$. $\dashv$

Note that since $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathrm{non}(\mathcal{M})$, $\mathfrak{b} = \mathfrak{s} = \aleph_1$ holds in the model $\mathbf{V}[G_{\omega_2}]$ constructed above.

Conjecture 7.19. *There are Shelah-Steprāns a.d. families in the model $\mathbf{V}[G_{\omega_2}]$ constructed in the proof of Theorem 7.18 (necessarily of size $\aleph_2$).*

A model with no Shelah-Steprāns a.d. families is constructed in [36]. This model is gotten by iterating posets of the form $\mathbb{L}(\mathcal{G})$, where $\mathcal{G}$ is some filter on a countable set. Since posets of this form always add dominating reals, $\mathfrak{b} > \aleph_1$ in the model in [36].

Question 7.20. *Is it consistent that there are no Shelah-Steprāns a.d. families and* $\mathrm{non}(\mathcal{M}) = \aleph_1$*?*

8. A MAD family that can not be extended to a Hurewicz ideal

We know that every Shelah-Steprāns MAD family is Hurewicz (Proposition 4.5). Non Canjar MAD families can be constructed in ZFC, see [10] or [15]). In order to solve the Problem 5.1, it would be enough to show that every MAD family can be extended to a Hurewicz ideal. Unfortunately, we will see that it is consistent with CH that this is not the case.

Definition 8.1. *(1) By Part we denote the set of partitions of ω into infinitely many infinite pieces.*

(2) Given $\mathcal{D} \in$ Part we define $\mathsf{fin}{\times}\mathsf{fin}(\mathcal{D}) = \{B \subseteq \omega \mid \forall^\infty D \in \mathcal{D}\,(|D \cap B| < \omega)\}$.

(3) Given two elements $\mathcal{D} = \{D(n) \mid n \in \omega\}, \mathcal{C} = \{C(n) \mid n \in \omega\}$ of Part, we say $\mathcal{C} \longrightarrow \mathcal{D}$ if one of the following conditions holds:

- *(a) $C(n)$ is almost disjoint with every $D(m)$ for almost all $n \in \omega$ or*
- *(b) there is $m \in \omega$ such that $C(n) \subseteq^* D(m)$ for almost all $n \in \omega$ or*
- *(c) for almost every $n \in \omega$ there is $m_n \in \omega$ such that $C(n) \subseteq^* D(m_n)$ and $m_k \neq m_r$ whenever $k \neq r$.*

Note that if $\mathcal{C} \longrightarrow \mathcal{D}$ then almost every element of $\mathcal{C}$ is in $\mathsf{fin}{\times}\mathsf{fin}(\mathcal{D})$. Now we define the following:

Definition 8.2. *Let $\mathbb{P}$ be the set of all $p = (\mathcal{A}_p, \mathcal{K}_p)$ such that $\mathcal{A}_p$ is a countable AD family and $\mathcal{K}_p$ is a countable subset of Part. We say that $p = (\mathcal{A}_p, \mathcal{K}_p) \leq q = (\mathcal{A}_q, \mathcal{K}_q)$ if the following conditions hold:*

(1) $\mathcal{A}_q \subseteq \mathcal{A}_p$ and $\mathcal{K}_q \subseteq \mathcal{K}_p$.

(2) *If* $\mathcal{C} \in \mathcal{K}_p \setminus \mathcal{K}_q$ *and* $\mathcal{D} \in \mathcal{K}_q$ *then* $\mathcal{C} \longrightarrow \mathcal{D}$.
(3) *If* $A \in \mathcal{A}_p \setminus \mathcal{A}_q$ *and* $\mathcal{D} \in \mathcal{K}_q$ *then* $A \in \mathsf{fin}\times\mathsf{fin}(\mathcal{D})$.

It is easy to see that $\mathbb{P}$ is a σ-closed forcing (so it does not add new reals). Let $\dot{\mathcal{A}}_{gen}$ be the name of $\bigcup\{\mathcal{A}_p \mid p \in \dot{G}\}$ (where $\dot{G}$ is the name for the generic filter). It is easy to see that $\dot{\mathcal{A}}_{gen}$ is forced to be an almost disjoint family, and we will see that $\dot{\mathcal{A}}_{gen}$ is forced to be a MAD family that can not be extended to a Hurewicz ideal. Recall that a family $\mathcal{H} \subseteq [\omega]^{\omega}$ is open dense in $\wp(\omega)/\mathsf{fin}$ if for every $A \in [\omega]^{\omega}$ there is a $B \in \mathcal{H}$ such that $B \subseteq^* A$ and $\mathcal{H}$ is closed under almost inclusion. It is well known and easy to see that every tall ideal on ω is open dense in $\wp(\omega)/\mathsf{fin}$ and the intersection of countably many open dense sets is open dense.

Lemma 8.3. *$\dot{\mathcal{A}}_{gen}$ is forced to be a MAD family.*

Proof. Letting $p = (\mathcal{A}_p, \mathcal{K}_p) \in \mathbb{P}$ and $X \in [\omega]^{\omega}$, we must find $q = (\mathcal{A}_q, \mathcal{K}_q) \leq p$ such that X is not AD with $\mathcal{A}_q$ (this is enough since $\mathbb{P}$ does not add new reals). Assume that X is AD with $\mathcal{A}_p$. By the previous remarks, we can find $A \subseteq^* X$ such that $A \in \mathsf{fin}\times\mathsf{fin}(\mathcal{D})$ for every $\mathcal{D} \in \mathcal{K}_p$. It is clear that $q = (\mathcal{A}_p \cup \{A\}, \mathcal{K}_p)$ has the desired properties. $\dashv$

Recall that an ideal $\mathcal{I}$ in a countable set is a P^+-ideal if for every decreasing family $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}^+$ there is $X \in \mathcal{I}^+$ such that $X \subseteq^* X_n$ for every $n \in \omega$. We will need the following:

Lemma 8.4. *Let $p = (\mathcal{A}_p, \mathcal{K}_p) \in \mathbb{P}$, $\dot{\mathcal{J}}$ be a name for a P^+-ideal such that $p \Vdash$ "$\dot{\mathcal{A}}_{gen} \subseteq \dot{\mathcal{J}}$" and $X \in [\omega]^{\omega}$ such that $p \Vdash$ "$X \in \dot{\mathcal{J}}^+$". There are $q \leq p$ and $Y \in [X]^{\omega}$ such that the following conditions hold:*

(1) *$Y \in \mathcal{A}_p^{\perp}$.*
(2) *For every $\mathcal{D} \in \mathcal{K}_p$, either Y is AD with all elements of $\mathcal{D}$ or there is $D \in \mathcal{D}$ such that $Y \subseteq^* D$.*
(3) *$q \Vdash$ "$Y \in \dot{\mathcal{J}}^+$".*

Proof. We first note that if $\mathcal{I}$ is a P^+-ideal, $Z \in \mathcal{I}^+$ and $\mathcal{C} = \{C(n) \mid n \in \omega\} \in \mathsf{Part}$, then there is $W \in [Z]^{\omega} \cap \mathcal{I}^+$ such that either W is AD with $\mathcal{C}$ or there is $C \in \mathcal{C}$ such that $W \subseteq C$. Indeed, if there is $n \in \omega$ such that $Z \cap C(n) \in \mathcal{I}^+$ then we define $W = Z \cap C(n)$, and if this is not the case, then $\{Z \setminus \bigcup_{i \leq n} C(i) \mid n \in \omega\} \subseteq \mathcal{I}^+$ forms a decreasing sequence, and we just let $W \subseteq Z$ be a pseudointersection in $\mathcal{I}^+$.

To prove the lemma, note that if $A \in \mathcal{A}_p$, then p forces that $X \setminus A$ is in $\dot{\mathcal{J}}^+$. Since both $\mathcal{A}_p$ and $\mathcal{K}_p$ are countable, the result then follows by the previous remarks. $\dashv$

With the lemma, we can now prove the following:

Lemma 8.5. *Let $p = (\mathcal{A}_p, \mathcal{K}_p) \in \mathbb{P}$, $\dot{\mathcal{J}}$ be a name for a P^+-ideal such that $p \Vdash$ "$\dot{\mathcal{A}}_{gen} \subseteq \dot{\mathcal{J}}$" and $\{X_n \mid n \in \omega\} \subseteq [\omega]^{\omega}$ such that $X_n \cap X_m = \emptyset$ whenever $n \neq m$ and $p \Vdash$ "$X_n \in \dot{\mathcal{J}}^+$" for every $n \in \omega$. There are $q = (\mathcal{A}_q, \mathcal{K}_q) \leq p$, $W \in [\omega]^{\omega}$ and $\{Y_n \mid n \in W\} \in \mathsf{Part}$ such that the following conditions hold:*

(1) *$Y_n \subseteq^* X_n$ for every $n \in W$.*
(2) *$\mathcal{Y} = \{Y_n \mid n \in W\} \in \mathcal{K}_q$ and $\mathcal{Y} \longrightarrow \mathcal{D}$ for every $\mathcal{D} \in \mathcal{K}_q \setminus \{\mathcal{Y}\}$.*
(3) *$q \Vdash$ "$Y_n \in \dot{\mathcal{J}}^+$" for every $n \in W$.*
(4) *q forces that every element in $\dot{\mathcal{A}}_{gen}$ has infinite intersection with only finitely many elements in $\mathcal{Y}$.*

Proof. Using the previous lemma, we recursively construct a sequence $\langle (p_n, Z_n) \rangle_{n\in\omega}$ with the following properties:

(1) $\langle p_n \rangle_{n\in\omega}$ is decreasing and $p_0 \leq p$.

(2) $Z_n \in [X_n]^{\omega}$.
(3) $p_n \Vdash$ "$Z_n \in \dot{\mathcal{J}}^{+}$".
(4) $Z_n \in \mathcal{A}_{p_{n-1}}^{\perp}$ (where $p_{-1} = p$).
(5) For every $\mathcal{D} \in \mathcal{K}_{p_{n-1}}$ either Z_n is AD with $\mathcal{D}$ or there is $D \in \mathcal{D}$ such that $Z_n \subseteq^* D$.

The construction is straightforward. Let $p_\omega = (\mathcal{A}_{p_\omega}, \mathcal{K}_{p_\omega})$ where $\mathcal{A}_{p_\omega} = \bigcup \mathcal{A}_{p_n}$ and $\mathcal{K}_{p_\omega} = \bigcup \mathcal{K}_{p_n}$. By our construction, p_ω has the following properties:

(1) If $A \in \mathcal{A}_{p_\omega}$ then $A \cap Z_n$ is finite for almost every $n \in \omega$.
(2) If $\mathcal{D} \in \mathcal{K}_{p_\omega}$ then for almost all $n \in \omega$, either Z_n is AD with $\mathcal{D}$ or there is $D \in \mathcal{D}$ such that $Z_n \subseteq^* D$.

We can then find $W \in [\omega]^{\omega}$ and $\mathcal{Y} = \{Y_n \mid n \in W\}$ such that the following conditions hold:

(1) $Y_n =^* Z_n$ for every $n \in W$.
(2) $\mathcal{Y} \in \mathsf{Part}$.
(3) $\mathcal{Y} \longrightarrow \mathcal{D}$ for every $\mathcal{D} \in \mathcal{K}_{p_\omega}$.

Such W can be found since its construction only requires intersecting countably many open dense subsets of $\wp(\omega) / \mathsf{fin}$. Letting $q = (\mathcal{A}_{p_\omega}, \mathcal{K}_{p_\omega} \cup \{\mathcal{Y}\})$, it is easy to see that q has the desired properties. $\dashv$

It is well known that every Canjar ideal is a P^+-ideal (see [21], [10] or [15]). We now have the following result, which is the heart of the construction:

Proposition 8.6. *Let $G \subseteq \mathbb{P}$ be a generic filter. The following holds in $V[G]$: If $\mathcal{J}$ is a Canjar ideal such that $\mathcal{A}_{gen} \subseteq \mathcal{J}$, then there are no $\{X_n \mid n \in \omega\} \subseteq \mathcal{J}^+$ such that $X_n \cap X_m = \emptyset$ for every $n \in \omega$.*

Proof. Let $p \in \mathbb{P}$, $\dot{\mathcal{J}}$ a name for a P^+-ideal extending $\dot{\mathcal{A}}_{gen}$ and $\{X_n \mid n \in \omega\}$ a pairwise disjoint family such that $p \Vdash$ "$\{X_n \mid n \in \omega\} \subseteq \dot{\mathcal{J}}^{+}$". We will find an extension of p that forces that $\dot{\mathcal{J}}$ is not Canjar. By the previous lemma, let $q = (\mathcal{A}_q, \mathcal{K}_q) \leq p$, $W \in [\omega]^{\omega}$ and $\{Y_n \mid n \in \omega\} \subseteq [\omega]^{\omega}$ be such that the following conditions hold:

(1) $Y_n \subseteq^* X_n$ for every $n \in W$.
(2) $\mathcal{Y} = \{Y_n \mid n \in W\} \in \mathcal{K}_q$ and $\mathcal{Y} \longrightarrow \mathcal{D}$ for every $\mathcal{D} \in \mathcal{K}_q \setminus \{\mathcal{Y}\}$.
(3) $q \Vdash$ "$Y_n \in \dot{\mathcal{J}}^{+}$" for every $n \in W$.
(4) q forces that every element in $\dot{\mathcal{A}}_{gen}$ has infinite intersection with only finitely many elements in $\mathcal{Y}$.

Let $W = \{w_n \mid n \in \omega\}$. For every $n \in \omega$, we define F_n to be the set of all $\{a_0, ..., a_n\}$ such that $a_0 < a_1 < ... < a_n$ and $a_i \in Y_{w_i}$ for every $i \leq n$. It is easy to see that $q \Vdash$ "$F_n \in (\dot{\mathcal{J}}^{<\omega})^{+}$" for every $n \in \omega$. We claim that q forces that $\langle F_n \mid n \in \omega \rangle$ witnesses that (in the extension) $\dot{\mathcal{J}}$ is not a Canjar ideal. It is enough to prove the following: For every $q_1 \leq q$ and $\langle H_n \rangle_{n \in \omega}$ such that $H_n \in [F_n]^{<\omega}$, there is $q_2 = (\mathcal{A}_{q_2}, \mathcal{K}_{q_2}) \leq q_1$ and $A \in \mathcal{I}(\mathcal{A}_{q_2})$ such that A has non empty intersection with every element of H_n for every $n \in \omega$ (recall that $\dot{\mathcal{J}}$ is forced to extend $\dot{\mathcal{A}}_{gen}$).

Let $q_1 = (\mathcal{A}_{q_1}, \mathcal{K}_{q_1})$ be an extension of q and let $\langle H_n \rangle_{n \in \omega}$ be such that $H_n \in [F_n]^{<\omega}$. We fix the following items:

(1) Let $\mathcal{A}_{q_1} = \{A_n \mid n \in \omega\}$ and define $B_n = \bigcup\limits_{i \leq n} A_i$ for every $n \in \omega$.
(2) Let $\mathcal{L} = \{\mathcal{C} \in \mathcal{K}_{q_1} \mid \mathcal{Y} \longrightarrow \mathcal{C}\}$.
    (a) Let $\mathcal{L}_{\mathsf{AD}}$ be the family of all $\mathcal{C} \in \mathcal{L}$ such that almost every element of $\mathcal{Y}$ is AD with $\mathcal{C}$. Fix an enumeration $\mathcal{L}_{\mathsf{AD}} = \{\mathcal{C}_n^{\mathsf{AD}} \mid n \in \omega\}$ where $\mathcal{C}_n^{\mathsf{AD}} = \{C_n^{\mathsf{AD}}(m) \mid m \in \omega\}$. Let $\overline{C}_n^{\mathsf{AD}}(m) = C_n^{\mathsf{AD}}(0) \cup ... \cup C_n^{\mathsf{AD}}(m)$.
    (b) Let $\mathcal{L}_{=}$ be the family of all $\mathcal{C} \in \mathcal{L}$ such that there is $C \in \mathcal{C}$ such that almost all elements of $\mathcal{Y}$ are almost contained in C. Fix an enumeration

$\mathcal{L}_= = \{\mathcal{C}^=_n \mid n \in \omega\}$ where $\mathcal{C}^=_n = \{C^=_n(m) \mid m \in \omega\}$ and let $c^=_n \in \omega$ be such that almost every element of $\mathcal{Y}$ is almost contained in $C^=_n(c^=_n)$.

(c) Let $\mathcal{L}_{\neq}$ be the family of all $\mathcal{C} \in \mathcal{L}$ such that for almost all $n \in W$, the Y_n are almost contained in pairwise distinct members of $\mathcal{C}$. Fix an enumeration $\mathcal{L}_{\neq} = \{\mathcal{C}^{\neq}_n \mid n \in \omega\}$ where $\mathcal{C}^{\neq}_n = \{C^{\neq}_n(m) \mid m \in \omega\}$.

Note that $\mathcal{L}$ is the disjoint union of $\mathcal{L}_{\mathsf{AD}}$, $\mathcal{L}_=$, and $\mathcal{L}_{\neq}$.

(3) Let $\mathcal{R} = \{\mathcal{D} \in \mathcal{K}_{q_1} \mid \mathcal{D} \longrightarrow \mathcal{Y}\}$.

(a) Let $\mathcal{R}_{\mathsf{AD}}$ be the family of all $\mathcal{D} \in \mathcal{R}$ such that almost every element of $\mathcal{D}$ is AD with $\mathcal{Y}$. Fix an enumeration $\mathcal{R}_{\mathsf{AD}} = \{\mathcal{D}^{\mathsf{AD}}_n \mid n \in \omega\}$ where $\mathcal{D}^{\mathsf{AD}}_n = \{D^{\mathsf{AD}}_n(m) \mid m \in \omega\}$. Let $d^{\mathsf{AD}}_n \in \omega$ such that if $d^{\mathsf{AD}}_n \leq m$ then $D^{\mathsf{AD}}_n(m)$ is AD with $\mathcal{Y}$ and let $\overline{D}^{\mathsf{AD}}_n(m) = \bigcup\{D^{\mathsf{AD}}_n(i) \mid d^{\mathsf{AD}}_n \leq i \leq m\}$.

(b) Let $\mathcal{R}_=$ be the family of all $\mathcal{D} \in \mathcal{R}$ such there is $Y_n \in \mathcal{Y}$ such that almost all elements of $\mathcal{D}$ are almost contained in Y_n. Fix an enumeration $\mathcal{R}_= = \{\mathcal{D}^=_n \mid n \in \omega\}$ where $\mathcal{D}^=_n = \{D^=_n(m) \mid m \in \omega\}$, and let $d^=_n, e^=_n \in \omega$ such that $D^=_n(m) \subseteq^* Y_{d^=_n}$ for every $m \geq e^=_n$.

(c) Let $\mathcal{R}_{\neq}$ be the family of all $\mathcal{D} \in \mathcal{R}$ such that almost all $D \in \mathcal{D}$ are almost contained in pairwise distinct members of $\mathcal{Y}$. Fix an enumeration $\mathcal{R}_{\neq} = \{\mathcal{D}^{\neq}_n \mid n \in \omega\}$ where $\mathcal{D}^{\neq}_n = \{D^{\neq}_n(m) \mid m \in \omega\}$.

Note that $\mathcal{R}$ is the disjoint union of $\mathcal{R}_{\mathsf{AD}}$, $\mathcal{R}_=$, and $\mathcal{R}_{\neq}$.

(4) Let $h : \omega \longrightarrow \omega$ be an increasing function such that for every $n \in \omega$, the following conditions hold:

(a) If $h(n) \leq m$ then B_n is almost disjoint with Y_{w_m}.
(b) If $h(n) \leq m$ then Y_{w_m} is almost disjoint with every element of $\mathcal{C}^{\mathsf{AD}}_n$.
(c) If $h(n) \leq m$ then $Y_{w_m} \subseteq^* C^=_n(c^=_n)$.
(d) If $h(n) \leq m < k$ then Y_{w_m} and Y_{w_k} are almost subsets of different elements in $\mathcal{C}^{\neq}_n$.
(e) If $h(n) \leq m$ then $D^{\mathsf{AD}}_n(m)$ is AD with $\mathcal{Y}$ (i.e. $d^{\mathsf{AD}}_n \leq h(n)$).
(f) If $h(n) \leq m$ then $D^=_n(m) \subseteq^* Y_{d^=_n}$ (i.e. $e^=_n \leq h(n)$).
(g) If $h(n) \leq m < k$ then $D^{\neq}_n(m)$ and $D^{\neq}_n(k)$ are almost subsets of different elements in $\mathcal{Y}$.

Note that $\mathcal{K}_{q_1} = \mathcal{L} \cup \mathcal{R} \cup \{\mathcal{Y}\}$. This follows from the construction of $\mathcal{Y}$ and $q_1 \leq q$.

We will say that a finite sequence $\langle (P_0, s_0), ..., (P_n, s_n)\rangle$ is *suitable* if the following conditions hold:

(1) $P_0, ..., P_n$ are non-empty finite consecutive intervals of ω.
(2) $\min(P_0) = 0$.
(3) $s_0 = P_0 \cap \left(\bigcup\left\{Y_{w_j} \mid j < h(0)\right\}\right)$.
(4) $s_{i+1} = P_{i+1} \cap \left(\bigcup\left\{Y_{w_j} \mid h(i) \leq j < h(i+1)\right\}\right)$.
(5) $s_i \neq \emptyset$ for every $i \leq n$.
(6) If $m < h(n)$ then $\bigcup\limits_{i \leq n} s_i$ has non-empty intersection with every element of H_m.
(7) $B_i \cap \left(\bigcup\left\{Y_{w_j} \mid h(i) \leq j < h(i+1)\right\}\right) \subseteq \max(P_i)$.
(8) $\overline{C}^{\mathsf{AD}}_l(h(i)) \cap \left(\bigcup\left\{Y_{w_j} \mid h(i) \leq j < h(i+1)\right\}\right) \subseteq \max(P_i)$ for every $l \leq i$.
(9) $\bigcup\left\{Y_{w_j} \mid h(i) \leq j < h(i+1)\right\} \setminus C^=_l(c^=_l) \subseteq \max(P_i)$ for every $l \leq i$.
(10) For every j such that $h(i) \leq j < h(i+1)$, if Y_{w_j} is not an almost subset of $C^{\neq}_k(l)$ with $k, l < i$ then $Y_{w_j} \cap C^{\neq}_k(l) \subseteq \max(P_i)$.
(11) $\overline{D}^{\mathsf{AD}}_l(h(i)) \cap \left(\bigcup\left\{Y_{w_j} \mid h(i) \leq j < h(i+1)\right\}\right) \subseteq \max(P_i)$ for every $l \leq i$.
(12) If $l \leq i$ and $e^=_l \leq j \leq h(i)$ then $D^=_l(j) \setminus \max(P_i) \subseteq Y_{d^=_l}$.
(13) For every j such that $h(i) \leq j < h(i+1)$, if $D^{\neq}_k(l)$ is not an almost subset of Y_{w_j} with $k, l < i$ then $Y_{w_j} \cap D^{\neq}_k(l) \subseteq \max(P_i)$.

Although the list of requirements needed to be verified is excessively long, it is not hard to see that for every suitable $\langle (P_0, s_0), ..., (P_n, s_n) \rangle$ there is (P_{n+1}, s_{n+1}) such that $\langle (P_0, s_0), ..., (P_n, s_n), (P_{n+1}, s_{n+1}) \rangle$ is suitable. Therefore, we can recursively construct $\langle (P_0, s_0), ..., (P_n, s_n), ... \rangle_{n\in\omega}$ such that every initial segment is suitable. Letting $A = \bigcup_{n\in\omega} s_n$ it is easy to see that $q_2 = (\mathcal{A}_{q_1} \cup \{A\}, \mathcal{K}_{q_1})$ is a condition extending q_1, which is the extension we were looking for. $\dashv$

We now obtain the main result of this section:

Theorem 8.7. *Let $G \subseteq \mathbb{P}$ be a generic filter. If $\mathcal{J}$ is a Canjar ideal extending $\mathcal{I}(\mathcal{A}_{gen})$ in $V[G]$, then the following hold:*

1. *For every $X \in \mathcal{J}^+$ there is $Y \in \mathcal{J}^+ \cap [X]^\omega$ such that $\mathcal{J}^* \restriction Y$ is an ultrafilter.*
2. *Forcing with $\mathbb{M}(\mathcal{J})$ diagonalizes an ultrafilter.*
3. *$\mathcal{J}$ is not a Hurewicz ideal.*

Proof. It is easy to see that 2 follows from 1 and 3 is a consequence of 2 (this follows e.g. from Proposition 5.3), so we only prove 1. Assume this is not the case, so for every $Y \in \mathcal{J}^+ \cap [X]^\omega$ it is the case that $\mathcal{J}^* \restriction Y$ is not an ultrafilter. We can then recursively construct a pairwise disjoint family $\{X_n \mid n \in \omega\} \subseteq \mathcal{J}^+ \cap [X]^\omega$, which contradicts the previous result. $\dashv$

We point out that under the Continuum Hypothesis, every MAD family can be extended to the dual of a Canjar ultrafilter [4]. Finally, we will prove the following result:

Proposition 8.8. *$\dot{\mathcal{A}}_{gen}$ is forced to be a Cohen-destructible MAD family.*

Proof. Recall that every Cohen indestructible MAD family has a restriction that is tight (see [20]). In fact, we will prove that no restriction of $\mathcal{A}_{gen}$ is weakly tight. Let $p \in \mathbb{P}$ and $X \in [\omega]^\omega$ such that $p \Vdash$ "$X \in \mathcal{I}(\dot{\mathcal{A}}_{gen})^+$". Since the ideals generated by MAD families are hereditarily meager, we may assume there is a pairwise disjoint family $\{X_n \mid n \in \omega\} \subseteq [X]^\omega$ such that $p \Vdash$ "$X_n \in \mathcal{I}(\dot{\mathcal{A}}_{gen})^+$" for every $n \in \omega$. Since the ideals generated by AD families are P^+-ideals (see [27] or [18]), we know by Lemma 8.5 there are $q \leq p$, $W \in [\omega]^\omega$ and $\{Y_n \mid n \in W\} \subseteq [\omega]^\omega$ such that the following conditions hold:

(1) $Y_n \subseteq^* X_n$ for every $n \in W$.
(2) $q \Vdash$ "$Y_n \in \mathcal{I}(\dot{\mathcal{A}}_{gen})^+$" for every $n \in W$.
(3) q forces that every element in $\dot{\mathcal{A}}_{gen}$ has infinite intersection with only finitely many elements in $\mathcal{Y}$.

Clearly q forces that the family $\{Y_n \mid n \in W\}$ witnesses that $\dot{\mathcal{A}}_{gen}$ is not weakly tight. $\dashv$

The result raises the following question:

Problem 8.9. *Is there (consistently) a Cohen indestructible MAD family that can not be extended to a Hurewicz ideal?*

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(Brendle) Graduate School of System Informatics, Kobe University, Rokkodai 1-1, Nada, Kobe 657-8501, Japan
Email address: brendle@kobe-u.ac.jp

(Guzmán) Centro de Ciencias Matemáticas, UNAM, Campus Morelia, 58089, México
Email address: oguzman@matmor.unam.mx
URL: https://www.matmor.unam.mx/~oguzman/

(Hrušák) Centro de Ciencias Matemáticas, UNAM, Campus Morelia, 58089, México
Email address: michael@matmor.unam.mx
URL: https://www.matmor.unam.mx/~michael/

(Raghavan) Department of Mathematics, National University of Singapore, Singapore 119076.
Email address: dilip.raghavan@protonmail.com
URL: https://dilip-raghavan.github.io/