LOWER BOUNDS OF SETS OF P-POINTS

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Abstract. We show that MA_κ implies that each collection of $P_\mathfrak{c}$-points of size at most κ which has a $P_\mathfrak{c}$-point as an RK upper bound also has a $P_\mathfrak{c}$-point as an RK lower bound.

1. Introduction

The Rudin-Keisler (RK) ordering of ultrafilters has received considerable attention since its introduction in the 1960s. For example, one can take a look at [10, 8, 9, 2, 4, 6, 5], or [7]. Recall the definition of the Rudin-Keisler ordering.

Definition 1. Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on ω. We say that $\mathcal{U} \leq_{RK} \mathcal{V}$ if there is a function f in ω^ω such that $A \in \mathcal{U}$ if and only if $f^{-1}(A) \in \mathcal{V}$ for every $A \subseteq \omega$.

When $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on ω and $\mathcal{U} \leq_{RK} \mathcal{V}$, we say that $\mathcal{U}$ is *Rudin-Keisler (RK) reducible* to $\mathcal{V}$, or that $\mathcal{U}$ is *Rudin-Keisler (RK) below* $\mathcal{V}$. In case $\mathcal{U} \leq_{RK} \mathcal{V}$ and $\mathcal{V} \leq_{RK} \mathcal{U}$ both hold, then we say that $\mathcal{U}$ and $\mathcal{V}$ are *Rudin-Keisler equivalent*, and write $\mathcal{U} \equiv_{RK} \mathcal{V}$.

Very early in the investigation of this ordering of ultrafilters, it was noticed that the class of P-points is particularly interesting. Recall that an ultrafilter $\mathcal{U}$ on ω is called a *P-point* if for any $\{a_n : n < \omega\} \subseteq \mathcal{U}$ there is an $a \in \mathcal{U}$ such that $a \subseteq^* a_n$ for every $n < \omega$, i.e. the set $a \setminus a_n$ is finite for every $n < \omega$. P-points were first constructed by Rudin in [10], under the assumption of the Continuum Hypothesis. The class of P-points forms a downwards closed initial segment of the class of all ultrafilters. In other words, if $\mathcal{U}$ is a P-point and $\mathcal{V}$ is any ultrafilter on ω with $\mathcal{V} \leq_{RK} \mathcal{U}$, then $\mathcal{V}$ is also a P-point. Hence understanding the order-theoretic structure of the class of P-points can provide information about the order-theoretic structure of the class of all ultrafilters on ω. One of the first systematic explorations of the order-theoretic properties of the class of all ultrafilters, and particularly of the class of P-points, under $\leq_{RK}$ was made by Blass in [3] and [2], where he proved many results about this ordering under the assumption of Martin's Axiom (MA).

Let us note here that it is not possible to construct P-points in ZFC only, as was proved by Shelah (see [11]). Thus some set-theoretic assumption is needed to ensure the existence of P-points. The most commonly used assumption when studying the order-theoretic properties of the class of P-points is MA. Under MA every ultrafilter has character $\mathfrak{c}$. Therefore, the $P_\mathfrak{c}$-points are the most natural class of P-points to focus on under MA. Again, the $P_\mathfrak{c}$-points form a downwards closed subclass of the P-points.

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Definition 2. An ultrafilter $\mathcal{U}$ on ω is called a *$P_{\mathfrak{c}}$-point* if for every $\alpha < \mathfrak{c}$ and any $\{a_i : i < \alpha\} \subseteq \mathcal{U}$ there is an $a \in \mathcal{U}$ such that $a \subseteq^* a_i$ for every $i < \alpha$.

In Theorem 5 from [2], Blass proved in ZFC that if $\{\mathcal{U}_n : n < \omega\}$ is a countable collection of P-points and if there is a P-point $\mathcal{V}$ such that $\mathcal{U}_n \leq_{RK} \mathcal{V}$ for every $n < \omega$, then there is a P-point $\mathcal{U}$ such that $\mathcal{U} \leq_{RK} \mathcal{U}_n$ for every $n < \omega$. In other words, if a countable family of P-points has an upper bound, then it also has a lower bound.

The main result of this paper generalizes Blass' theorem to families of $P_{\mathfrak{c}}$-points of size less than $\mathfrak{c}$ under MA. More precisely, if MA holds and a family of $P_{\mathfrak{c}}$-points of size less than $\mathfrak{c}$ has an RK upper bound which is a $P_{\mathfrak{c}}$-point, then the family also has an RK lower bound.

Blass proved his result via some facts from [1] about non-standard models of complete arithmetic. In order to state these results, we introduce a few notions from [1]. The language L will consist of symbols for all relations and all functions on ω. Let N be the standard model for this language, its domain is ω and each relation or function denotes itself. Let M be an elementary extension of N, and let ${}^*R$ be the relation in M denoted by R, and let ${}^*f$ be the function in M denoted by f. Note that if $a \in M$, then the set $\{{}^*f(a) : f : \omega \to \omega\}$ is the domain of an elementary submodel of M. Submodel like this, i.e. generated by a single element, will be called *principal*. It is not difficult to prove that a principal submodel generated by a is isomorphic to the ultrapower of the standard model by the ultrafilter $\mathcal{U}_a = \{X \subseteq \omega : a \in {}^*X\}$. If $A, B \subseteq M$, we say that they are *cofinal with each other* iff $(\forall a \in A)(\exists b \in B)\ a\ {}^*{\leq}\ b$ and $(\forall b \in B)(\exists a \in A)\ b\ {}^*{\leq}\ a$. Finally, we can state Blass' theorem.

Theorem 3 (Blass, Theorem 3 in [1])**.** *Let M_i $(i < \omega)$ be countably many pairwise cofinal submodels of M. Assume that at least one of the M_i is principal. Then $\bigcap_{i<\omega} M_i$ is cofinal with each M_i, in fact it contains a principal submodel cofinal with each M_i.*

After proving this theorem, Blass states that it is not known to him whether Theorem 3 can be extended to larger collections of submodels. The proof of our main result clarifies this, namely in Theorem 19 below we prove that under MA it is possible to extend it to collections of models of size less than $\mathfrak{c}$ provided that there is a principal model that is isomorphic to an ultrapower induced by a $P_{\mathfrak{c}}$-point. Then we proceed and use this result to prove Theorem 20 where we extend Theorem 5 from [2] to collections of fewer than $\mathfrak{c}$ many $P_{\mathfrak{c}}$-points.

Recall that MA_α is the statement that for every partial order P which satisfies the countable chain condition and for every collection $\mathcal{D} = \{D_i : i < \alpha\}$ of dense subsets of P, there is a filter $G \subseteq P$ such that $G \cap D_i \neq \emptyset$ for every $i < \alpha$.

2. The lower bound

In this section we prove the results of the paper. We begin with a purely combinatorial lemma about functions.

Definition 4. Let α be an ordinal, let $\mathcal{F} = \{f_i : i < \alpha\} \subseteq \omega^\omega$ be a family of functions, and let A be a subset of α. We say that a set $F \subseteq \omega$ is *$(A, \mathcal{F})$-closed* if $f_i^{-1}(f_i'' F) \subseteq F$ for each $i \in A$.

Remark 5. Notice that if F is $(A, \mathcal{F})$-closed, then $f_i^{-1}(f_i'' F) = F$ for each $i \in A$.

Lemma 6. *Let α be an ordinal, let $\mathcal{F} = \{f_i : i < \alpha\} \subseteq \omega^\omega$ be a family of functions, and let A be a subset of α. Suppose that $m < \omega$, and that F_k is $(A, \mathcal{F})$-closed subset of ω, for each $k < m$. Then the set $F = \bigcup_{k<m} F_k$ is $(A, \mathcal{F})$-closed.*

Proof. To prove that F is $(A, \mathcal{F})$-closed take any $i \in A$, and $n \in f_i^{-1}(f_i'' F)$. This means that there is some $n' \in F$ such that $f_i(n) = f_i(n')$. Let $k < m$ be such that $n' \in F_k$. Then $n \in f_i^{-1}(f_i'' F_k)$. Since F_k is $(A, \mathcal{F})$-closed, $n \in f_i^{-1}(f_i'' F_k) \subseteq F_k$. Thus $n \in F_k \subseteq F$. □

Lemma 7. *Let $\alpha < \mathfrak{c}$ be an ordinal. Let $\mathcal{F} = \{f_i : i < \alpha\} \subseteq \omega^\omega$ be a family of finite-to-one functions. Suppose that for each $i, j < \alpha$ with $i < j$, there is $l < \omega$ such that $f_j(n) = f_j(m)$ whenever $f_i(n) = f_i(m)$ and $n, m \geq l$. Then for each finite $A \subseteq \alpha$, and each $n < \omega$, there is a finite $(A, \mathcal{F})$-closed set F such that $n \in F$.*

Proof. First, if A is empty, then we can take $F = \{n\}$. So fix a non-empty finite $A \subseteq \alpha$, and $n < \omega$. For each $i, j \in A$ such that $i < j$, by the assumption of the lemma, take $l_{ij} < \omega$ such that for each $n, m \geq l_{ij}$, if $f_i(n) = f_i(m)$, then $f_j(n) = f_j(m)$. Since A is a finite set, there is $l = \max\{l_{ij} : i, j \in A,\ i < j\}$. So l has the property that for every $i, j \in A$ with $i < j$, if $f_i(n) = f_i(m)$ and $n, m \geq l$, then $f_j(n) = f_j(m)$.

Let $i_0 = \max(A)$. Clearly, $f_i'' l$ is finite for each $i \in A$, and since each f_i is finite-to-one the set $f_i^{-1}(f_i'' l)$ is finite for every $i \in A$. Since the set A is also finite, there is $l' < \omega$ such that $\bigcup_{i \in A} f_i^{-1}(f_i'' l) \subseteq l'$. Again, since f_{i_0} is finite-to-one there is $l'' < \omega$ such that $f_{i_0}^{-1}\left(f_{i_0}'' l'\right) \subseteq l''$. Note that by the definition of numbers l' and l'', we have $l'' \geq l' \geq l$.

Claim 8. *For all $k < \omega$, if $k \geq l''$, then the set $f_{i_0}^{-1}(f_{i_0}'' \{k\})$ is $(A, \mathcal{F})$-closed.*

Proof. Fix $k \geq l''$ and let $X = f_{i_0}^{-1}(f_{i_0}'' \{k\})$. First observe that $X \cap l' = \emptyset$. To see this suppose that there is $m \in X \cap l'$. Since $m \in X$, $f_{i_0}(m) = f_{i_0}(k)$. Together with $m \in l'$, this implies that $k \in f_{i_0}^{-1}(f_{i_0}'' \{m\}) \subseteq f_{i_0}^{-1}(f_{i_0}'' l') \subseteq l''$. Thus $k < l''$ contradicting the choice of k. Secondly, observe that if $m < l$ and $k' \in X$, then $f_i(m) \neq f_i(k')$ for each $i \in A$. To see this, fix $m < l$ and $k' \in X$, and suppose that for some $i \in A$, $f_i(m) = f_i(k')$. This means that $k' \in f_i^{-1}(f_i'' \{m\}) \subseteq f_i^{-1}(f_i'' l) \subseteq l'$ contradicting the fact that $X \cap l' = \emptyset$.

Now we will prove that X is $(A, \mathcal{F})$-closed. Take any $i \in A$ and any $m \in f_i^{-1}(f_i'' X)$. We should prove that $m \in X$. Since $m \in f_i^{-1}(f_i'' X)$, $f_i(m) \in f_i'' X$ so there is some $k' \in X$ such that $f_i(m) = f_i(k')$. By the second observation, $m \geq l$. By the first observation $k' \geq l' \geq l$. By the assumption of the lemma, since $m, k' \geq l$, and $f_i(m) = f_i(k')$, it must be that $f_{i_0}(m) = f_{i_0}(k')$. Since $k' \in X = f_{i_0}^{-1}(f_{i_0}'' \{k\})$, it must be that $f_{i_0}(k) = f_{i_0}(k') = f_{i_0}(m)$. This means that $m \in f_{i_0}^{-1}(f_{i_0}'' \{k\}) = X$ as required. Thus $f_{i_0}^{-1}(f_{i_0}'' \{k\})$ is $(A, \mathcal{F})$-closed. □

Now we inductively build a tree $T \subseteq \omega^{<\omega}$ we will be using in the rest of the proof. Fix a function $\Phi : \omega \to \omega^{<\omega}$ so that $\Phi^{-1}(\sigma)$ is infinite for each $\sigma \in \omega^{<\omega}$. For each $m < \omega$ let $u_m = \Phi(m)\,(|\Phi(m)| - 1)$, i.e. u_m is the last element of the sequence $\Phi(m)$. Let $T_0 = \{\emptyset, \langle n \rangle\}$ (recall that n is given in the statement of the lemma). Suppose that $m \geq 1$, and that T_m is given. If $\Phi(m)$ is a leaf node of T_m, then let

$$Z_m = \left(\bigcup_{i \in A} f_i^{-1}(f_i'' \{u_m\})\right) \setminus \left(\bigcup_{\eta \in T_m} \mathrm{range}(\eta)\right),$$

and $T_{m+1} = T_m \cup \{\Phi(m)^\frown \langle k\rangle : k \in Z_m\}$. If $\Phi(m)$ is not a leaf node of T_m, then $T_{m+1} = T_m$. Finally, let $T = \bigcup_{m<\omega} T_m$ and $F = \bigcup_{\eta\in T} \operatorname{range}(\eta)$.

Claim 9. *If σ is a non-empty element of the tree T, then there is $m_0 \geq 1$ such that σ is a leaf node of T_{m_0}, that $\sigma = \Phi(m_0)$ and that*

$$\bigcup_{i\in A} f_i^{-1}(f_i''\{u_{m_0}\}) \subseteq \bigcup_{\eta\in T_{m_0+1}} \operatorname{range}(\eta).$$

Proof. Fix a non-empty σ in T. Let $m_1 = \min\{k < \omega : \sigma \in T_k\}$. Since $|\sigma| > 0$, σ is a leaf node of T_{m_1}. Consider the set $W = \{m \geq m_1 : \Phi(m) = \sigma\}$. Since the set $\{m < \omega : \Phi(m) = \sigma\}$ is infinite, W is non-empty subset of positive integers, so it has a minimum. Let $m_0 = \min W$. Note that if $m_0 = m_1$, then σ is a leaf node of T_{m_0}. If $m_0 > m_1$, by the construction of the tree T, since $\Phi(k) \neq \sigma$ whenever $m_1 \leq k < m_0$, it must be that σ is a leaf node of every T_k for $m_1 < k \leq m_0$. Thus σ is a leaf node of T_{m_0} and $\Phi(m_0) = \sigma$. Again by the construction of the tree T, we have $T_{m_0+1} = T_{m_0} \cup \{\sigma^\frown \langle k\rangle : k \in Z_{m_0}\}$. This means that

$$\bigcup_{\eta\in T_{m_0+1}} \operatorname{range}(\eta) = Z_{m_0} \cup \bigcup_{\eta\in T_{m_0}} \operatorname{range}(\eta).$$

Finally, the definition of Z_{m_0} implies that

$$\bigcup_{i\in A} f_i^{-1}(f_i''\{u_{m_0}\}) \subseteq Z_{m_0} \cup \bigcup_{\eta\in T_{m_0}} \operatorname{range}(\eta) = \bigcup_{\eta\in T_{m_0+1}} \operatorname{range}(\eta),$$

as required. □

Claim 10. *The set F is $(A, \mathcal{F})$-closed, and contains n as an element.*

Proof. Since $\langle n\rangle \in T_0$, $n \in F$. To see that F is $(A, \mathcal{F})$-closed, take any $j \in A$, and any $w \in f_j^{-1}\left(f_j''F\right)$. We have to show that $w \in F$. Since $w \in f_j^{-1}\left(f_j''F\right)$, there is $m \in F$ such that $f_j(w) = f_j(m)$. Since $m \in F = \bigcup_{\eta\in T} \operatorname{range}(\eta)$, there is σ in T such that $\sigma(k) = m$ for some $k < \omega$. Consider $\sigma \restriction (k+1)$. Since $\sigma \restriction (k+1) \in T$, by Claim 9 there is $m_0 \geq 1$ such that $\Phi(m_0) = \sigma \restriction (k+1)$, that $\sigma \restriction (k+1)$ is a leaf node of T_{m_0} and that (note that $u_{m_0} = \sigma(k) = m$)

$$\bigcup_{i\in A} f_i^{-1}(f_i''\{m\}) \subseteq \bigcup_{\eta\in T_{m_0+1}} \operatorname{range}(\eta) \subseteq \bigcup_{\eta\in T} \operatorname{range}(\eta) = F.$$

So $w \in f_j^{-1}(f_j''\{m\}) \subseteq F$ as required. □

Claim 11. *The tree T is finite.*

Proof. First we prove that each level of T is finite. For $k < \omega$ let $T_{(k)}$ be the k-th level of T, i.e. $T_{(k)} = \{\sigma \in T : |\sigma| = k\}$. Clearly $T_{(0)}$ and $T_{(1)}$ are finite. So suppose that $T_{(k)}$ is finite. Let $T_{(k)} = \{\sigma_0, \sigma_2, \ldots, \sigma_t\}$ be enumeration of that level. For $s \leq t$ let m_s be such that $\Phi(m_s) = \sigma_s$ and that σ_s is a leaf node of T_{m_s}. Note that by the construction of the tree T all nodes at the level $T_{(k+1)}$ are of the form $\sigma_s^\frown \langle r\rangle$ where $s \leq t$ and $r \in Z_{m_s}$. Since the set A is finite and all functions f_i (for $i \in A$) are finite-to-one, Z_{m_s} is finite for every $s \leq t$. Thus there are only finitely many nodes of the form $\sigma_s^\frown \langle r\rangle$ where $s \leq t$ and $r \in Z_{m_s}$, hence the level $T_{(k+1)}$ must also be finite. This proves by induction that each level of T is finite.

Suppose now that T is infinite. By König's lemma, since each level of T is finite, T has an infinite branch b. By definition of the sets Z_m ($m < \omega$), each node of T is 1-1 function, so b is also an injection from ω into ω. In particular, the range of b is infinite. Let $k = \min\{m < \omega : b(m) \geq l''\}$, and let $\sigma = b \restriction (k+1)$. Clearly, $\sigma \in T$. By Claim 9, there is $m_0 < \omega$ such that σ is a leaf node of T_{m_0}, that $\Phi(m_0) = \sigma$, and that $\bigcup_{i\in A} f_i^{-1}(f_i''\{\sigma(k)\}) \subseteq \bigcup_{\eta\in T_{m_0+1}} \operatorname{range}(\eta)$. Since $\sigma(k) = b(k) \geq l''$, Claim

8 implies that the set $Y = f_{i_0}^{-1}(f_{i_0}'' \{\sigma(k)\})$ is $(A, \mathcal{F})$-closed. By the construction $T_{m_0+1} = T_{m_0} \cup \{\sigma^\frown \langle m \rangle : m \in Z_{m_0}\}$. Since b is an infinite branch, there is $m' \in Z_{m_0}$ such that $b(k+1) = m'$. Now $m' \in Z_{m_0} \subseteq \bigcup_{i \in A} f_i^{-1}(f_i'' \{\sigma(k)\})$, the fact that $\sigma(k) \in Y$, and the fact that Y is $(A, \mathcal{F})$-closed, together imply that

$$m' \in \bigcup_{i \in A} f_i^{-1}(f_i'' \{\sigma(k)\}) \subseteq \bigcup_{i \in A} f_i^{-1}(f_i'' Y) \subseteq Y.$$

Consider the node $\tau = \sigma^\frown \langle m' \rangle = b \restriction (k+2)$. Since b is an infinite branch, it must be that $\tau^\frown \langle b(k+2) \rangle \in T$. By Claim 9, there is m_1 such that τ is a leaf node of T_{m_1} and that $\Phi(m_1) = \tau$. Clearly, $m_1 > m_0$ and $\tau^\frown \langle b(k+2) \rangle \in T_{m_1+1}$. Recall that we have already shown that $\bigcup_{i \in A} f_i^{-1}(f_i'' \{\sigma(k)\}) \subseteq \bigcup_{\eta \in T_{m_0+1}} \mathrm{range}(\eta)$. Thus $Y \subseteq \bigcup_{\eta \in T_{m_0+1}} \mathrm{range}(\eta)$. This, together with the fact that $\tau(k+1) = m' \in Y$, that Y is $(A, \mathcal{F})$-closed, and $m_1 > m_0$ jointly imply that

$$\bigcup_{i \in A} f_i^{-1}(f_i'' \{\tau(k+1)\}) \subseteq Y \subseteq \bigcup_{\eta \in T_{m_0+1}} \mathrm{range}(\eta) \subseteq \bigcup_{\eta \in T_{m_1}} \mathrm{range}(\eta).$$

This means that

$$b(k+2) \in Z_{n_1} = \bigcup_{i \in A} f_i^{-1}(f_i'' \{\tau(k+1)\}) \setminus \bigcup_{\eta \in T_{m_1}} \mathrm{range}(\eta) = \emptyset,$$

which is clearly impossible. Thus, T is not infinite. □

To finish the proof, note that by Claim 10 the set F is $(A, \mathcal{F})$-closed and contains n as an element, while by Claim 11 the set F is finite. So F satisfies all the requirements of the conclusion of the lemma. □

The following lemma is the main application of Martin's Axiom. Again, it does not directly deal with ultrafilters, but with collections of functions.

Lemma 12 (MA_α)**.** *Let $\mathcal{F} = \{f_i : i < \alpha\} \subseteq \omega^\omega$ be a family of finite-to-one functions. Suppose that for each non-empty finite set $A \subseteq \alpha$, and each $n < \omega$, there is a finite $(A, \mathcal{F})$-closed set F containing n as an element. Then there is a finite-to-one function $h \in \omega^\omega$, and a collection $\{e_i : i < \alpha\} \subseteq \omega^\omega$ such that for each $i < \alpha$, there is $l < \omega$ such that $h(n) = e_i(f_i(n))$ whenever $n \geq l$.*

Proof. We will apply MA_α, so we first define the poset we will be using. Let $\mathcal{P}$ be the set of all $p = \langle g_p, h_p \rangle$ such that

(I) $h_p : N_p \to \omega$ where N_p is a finite subset of ω,
(II) $g_p = \langle g_p^i : i \in A_p \rangle$ where $A_p \in [\alpha]^{<\aleph_0}$, and $g_p^i : f_i'' N_p \to \omega$ for each $i \in A_p$,
(III) N_p is $(A_p, \mathcal{F})$-closed.

Define the ordering relation $\leq$ on $\mathcal{P}$ as follows: $q \leq p$ iff

(IV) $N_p \subseteq N_q$,
(V) $A_p \subseteq A_q$,
(VI) $h_q \restriction N_p = h_p$,
(VII) $g_q^i \restriction f_i'' N_p = g_p^i$ for each $i \in A_p$,
(VIII) $h_q(n) > h_q(m)$ whenever $m \in N_p$ and $n \in N_q \setminus N_p$,
(IX) $h_q(n) = g_q^i(f_i(n))$ for each $n \in N_q \setminus N_p$ and $i \in A_p$.

It is clear that $\langle \mathcal{P}, \leq \rangle$ is a partially ordered set.

Claim 13. *Let $p \in \mathcal{P}$, $n_0 < \omega$, and suppose that $A \subseteq \alpha$ is finite such that $A_p \subseteq A$. Then there is $q \leq p$ such that $n_0 \subseteq N_q$ and that N_q is $(A, \mathcal{F})$-closed.*

Proof. Applying the assumption of the lemma to the finite set A, and each $k \in n_0 \cup N_p$, we obtain sets F_k ($k \in n_0 \cup N_p$) such that $k \in F_k$ and $f_i^{-1}(f_i'' F_k) \subseteq F_k$ for each $k \in n_0 \cup N_p$ and $i \in A$. Let $N_q = \bigcup_{k \in n_0 \cup N_p} F_k$, let $A_q = A_p$, and let $t = \max\{h_p(k) + 1 : k \in N_p\}$. Finally, define

$$h_q(n) = \begin{cases} h_p(n), & \text{if } n \in N_p \\ t, & \text{if } n \in N_q \setminus N_p \end{cases} \quad \text{and } g_q^i(k) = \begin{cases} g_p^i(k), & \text{if } k \in f_i'' N_p \\ t, & \text{if } k \in f_i'' N_q \setminus f_i'' N_p \end{cases} \quad \text{for } i \in A_q.$$

Let g_q denote the sequence $\langle g_q^i : i \in A_q \rangle$. Clearly $n_0 \subseteq N_q$. By Lemma 6, N_q is $(A, \mathcal{F})$-closed. We still have to show that $q = \langle g_q, h_q \rangle \in \mathcal{P}$ and $q \leq p$. Since h_q is defined on N_q and N_q finite, since $A_q = A_p$ and g_p^i is defined on $f_i'' N_q$ for $i \in A_p$, and since N_q is $(A_q, \mathcal{F})$-closed, conditions (I)-(III) are satisfied by q. Thus $q \leq p$. Next we show $q \leq p$. Conditions (IV)-(VII) are obviously satisfied by the definition of q. Since $h_q(n) = t > h_p(k) = h_q(k)$ for each $n \in N_q \setminus N_p$ and $k \in N_p$, conditions (VII) is also satisfied. So we still have to check (IX). Take any $i \in A_p$ and $n \in N_q \setminus N_p$. By the definition of h_q, we have $h_q(n) = t$. Once we prove that $f_i(n) \in f_i'' N_q \setminus f_i'' N_p$, we will be done because in that case the definition of g_p^i implies that $g_p^i(f_i(n)) = t$ as required. So suppose the contrary, that $f_i(n) \in f_i'' N_p$. Since p is a condition and $A_q = A_p$, it must be that $n \in f_i^{-1}(f_i'' N_p) \subseteq N_p$. But this contradicts the choice of n. Thus condition (IX) is also satisfied and $q \leq p$. □

Claim 14. *Let $p \in \mathcal{P}$, and $j_0 < \alpha$. Then there is $q \leq p$ such that $j_0 \in A_q$.*

Proof. Let $A_q = A_p \cup \{j_0\}$. Applying Claim 13 to A_q and $n = 0$, we obtain a condition $p' \leq p$ such that $N_{p'}$ is $(A_q, \mathcal{F})$-closed. Let $N_q = N_{p'}$, $h_q = h_{p'}$, and $g_q^i = g_{p'}^i$ for $i \in A_p$. Define $g_q^{j_0}(k) = 0$ for each $k \in f_{j_0}'' N_{p'}$, and let g_p denote the sequence $\langle g_q^i : i \in A_q \rangle$. Since $j_0 \in A_q$, to finish the proof of the claim it is enough to show that $q = \langle g_q, h_q \rangle \in \mathcal{P}$, and that $q \leq p'$. Conditions (I)-(III) are clear from the definition of q because $p' \in \mathcal{P}$ and $N_q = N_{p'}$, $h_q = h_{p'}$, $g_q^{j_0} : f_{j_0}'' N_q \to \omega$, and $g_q^i = g_p^i$ for $i \in A_q \setminus \{j_0\}$. Conditions (IV)-(VII) are clear by the definition of h_q and g_q. Conditions (VIII) and (IX) are vacuously true because $N_{p'} = N_q$. Thus the claim is proved. □

Claim 15. *If $p, q \in \mathcal{P}$ are such that $h_p = h_q$ and $g_p^i = g_q^i$ for $i \in A_p \cap A_q$, then p and q are compatible in $\mathcal{P}$.*

Proof. We proceed to define $r \leq p, q$. Let $N = N_p = N_q$. Let

$$t = \max\{h_p(n) + 1 : n \in N\}.$$

Applying the assumption of the lemma to $A = A_p \cup A_q$, and each $k \in N$, we obtain $(A, \mathcal{F})$-closed sets F_k ($k \in N$). By Claim 6, the set $N_r = \bigcup_{k \in N} F_k$ is $(A, \mathcal{F})$-closed. Let $A_r = A$, and define

$$h_r(n) = \begin{cases} h_p(n), & \text{if } n \in N \\ t, & \text{if } n \in N_r \setminus N \end{cases} \text{and } g_r^i(k) = \begin{cases} g_p^i(k), & \text{if } i \in A_p \text{ and } k \in f_i'' N \\ g_q^i(k), & \text{if } i \in A_q \text{ and } k \in f_i'' N \\ t, & \text{if } k \in f_i'' N_r \setminus f_i'' N \end{cases},$$

for $i \in A_r$. Let g_r denote the sequence $\langle g_r^i : i \in A_r \rangle$. As we have already mentioned, $r = \langle h_r, g_r \rangle$ satisfies (III), and it clear that (I) and (II) are also true for r. To see that $r \leq p$ and $r \leq q$ note that conditions (IV)-(VIII) are clearly satisfied. We will check that r and p satisfy (IX) also. Take any $n \in N_r \setminus N$ and $i \in A_p$. By the definition of h_r, $h_r(n) = t$. By the definition of g_r^i, if $f_i(n) \in f_i'' N_r \setminus f_i'' N$, then

$g_r^i(f_i(n)) = t = h_r(n)$. So suppose this is not the case, i.e. that $f_i(n) \in f_i''N$. This would mean that $n \in f_i^{-1}(f_i''N)$, which is impossible because $n \notin N$ and N is $(A, \mathcal{F})$-closed because $p \in \mathcal{P}$. Thus we proved $r \leq p$. In the same way it can be shown that $r \leq q$. $\square$

Claim 16. *The poset $\mathcal{P}$ satisfies the countable chain condition.*

Proof. Let $\langle p_\xi : \xi < \omega_1 \rangle$ be an uncountable set of conditions in $\mathcal{P}$. Since $h_{p_\xi} \in [\omega \times \omega]^{<\omega}$ for each $\xi < \omega_1$, there is an uncountable set $\Gamma \subseteq \omega_1$, and $h \in [\omega \times \omega]^{<\omega}$ such that $h_{p_\xi} = h$ for each $\xi \in \Gamma$. Consider the set $\langle A_{p_\xi} : \xi \in \Gamma \rangle$. By the Δ-system lemma, there is an uncountable set $\Delta \subseteq \Gamma$, and a finite set $A \subseteq \alpha$ such that $A_{p_\xi} \cap A_{p_\eta} = A$ for each $\xi, \eta \in \Delta$. Since Δ is uncountable and $g_{p_\xi}^i \in [\omega \times \omega]^{<\omega}$ for each $i \in A$ and $\xi \in \Delta$, there is an uncountable set $\Theta \subseteq \Delta$ and g_i for each $i \in A$, such that $g_{p_\xi}^i = g_i$ for each $\xi \in \Theta$ and $i \in A$. Let ξ and η in Θ be arbitrary. By Claim 15, p_ξ and p_η are compatible in $\mathcal{P}$. $\square$

Consider sets $D_j = \{p \in \mathcal{P} : j \in A_p\}$ for $j \in \alpha$, and $D_m = \{p \in \mathcal{P} : m \in N_p\}$ for $m \in \omega$. By Claim 14 and Claim 13, these sets are dense in $\mathcal{P}$. By MA_α there is a filter $\mathcal{G} \subseteq \mathcal{P}$ intersecting all these sets. Clearly, $h = \bigcup_{p \in \mathcal{G}} h_p$ and $e_i = \bigcup_{p \in \mathcal{G}} g_p^i$, for each $i \in \alpha$, are functions from ω into ω. We will prove that these functions satisfy the conclusion of the lemma. First we will prove that h is finite-to-one. Take any $m \in \omega$ and let $k = h(m)$. By the definition of h, there is $p \in \mathcal{G}$ such that $h_p(m) = k$. Suppose that $h^{-1}(\{k\}) \nsubseteq N_p$. This means that there is an integer $m_0 \notin N_p$ such that $h(m_0) = k$. Let $q \in \mathcal{G}$ be such that $h_q(m_0) = k$. Now for a common extension $r \in \mathcal{G}$ of both p and q, it must be that $h_r(m_0) = h_p(m)$, contradicting the fact that $r \leq p$, in particular condition (VIII) is violated in this case. We still have to show that for each $i \in \alpha$, there is $l \in \omega$ such that $h(n) = e_i(f_i(n))$ whenever $n \geq l$. So take $i \in \alpha$. By Claim 14, there is $p \in \mathcal{G}$ such that $i \in A_p$. Let $l = \max(N_p) + 1$. We will prove that l is as required. Take any $n \geq l$. By Claim 13, there is $q \in \mathcal{G}$ such that $n \in q$. Let $r \in \mathcal{G}$ be a common extension of p and q. Since $n \notin N_p$ and $r \leq p$, it must be that $h_r(n) = g_r^i(f_i(n))$, according to condition (IX). Hence $h(n) = e_i(f_i(n))$, as required. $\square$

Before we move to the next lemma let us recall that if c is any element of the model M, then $\mathcal{U} = \{X \subseteq \omega : c \in {}^*X\}$ is ultrafilter on ω.

Lemma 17. *Let $\alpha < \mathfrak{c}$ be an ordinal. Let $\langle M_i : i < \alpha \rangle$ be a $\subseteq$-decreasing sequence of principal submodels of M, i.e. each M_i is generated by a single element a_i and $M_j \subseteq M_i$ whenever $i < j < \alpha$. Let each M_i $(i < \alpha)$ be cofinal with M_0. Suppose that $\mathcal{U}_0 = \{X \subseteq \omega : a_0 \in {}^*X\}$ is a $P_\mathfrak{c}$-point. Then there is a family $\{f_i : i < \alpha\} \subseteq \omega^\omega$ of finite-to-one functions such that ${}^*f_i(a_0) = a_i$ for $i < \alpha$, and that for $i, j < \alpha$ with $i < j$, there is $l < \omega$ such that $f_j(n) = f_j(m)$ whenever $f_i(n) = f_i(m)$ and $m, n \geq l$.*

Proof. Let $i < j < \alpha$. Since $M_j \subseteq M_i$, and M_i is generated by a_i, there is a function $\varphi_{ij} : \omega \to \omega$ such that ${}^*\varphi_{ij}(a_i) = a_j$. Since M_j is cofinal with M_i, by Lemma in [1, page 104], if $i < j < \alpha$, then there is a set $Y_{ij} \subseteq \omega$ such that $a_i \in {}^*Y_{ij}$ and that $\varphi_{ij} \restriction Y_{ij}$ is finite-to-one. For $i < \alpha$ let $g_i : \omega \to \omega$ be defined as follows: if $n < \omega$, then for $n \notin Y_{0i}$ let $g_i(n) = n$, while for $n \in Y_{0i}$ let $g_i(n) = \varphi_{0i}(n)$. Note that ${}^*g_i(a_0) = a_i$, and that g_i is finite-to-one. The latter fact follows since g_i is one-to-one on $\omega \setminus Y_{0i}$ and on Y_{0i} it is equal to φ_{0i}, which is finite-to-one on Y_{0i}. Now by the second part of Lemma on page 104 in [1], for

$i < j < \alpha$ there is a finite-to-one function $\pi_{ij} : \omega \to \omega$ such that ${}^*\varphi_{ij}(a_i) = {}^*\pi_{ij}(a_i)$. Note that this means that ${}^*g_j(a_0) = {}^*\pi_{ij}({}^*g_i(a_0))$ for $i < j < \alpha$, i.e. the set $X_{ij} = \{n \in \omega : g_j(n) = \pi_{ij}(g_i(n))\}$ is in $\mathcal{U}_0$. Since $\alpha < \mathfrak{c}$ and $\mathcal{U}_0$ is a $P_\mathfrak{c}$-point, there is a set $X \subseteq \omega$ such that $X \in \mathcal{U}_0$ and that the set $X \setminus X_{ij}$ is finite whenever $i < j < \alpha$. Consider the sets $W_i = g_i'' X$ for $i < \alpha$. For each $i < \alpha$, let $W_i = W_i^0 \cup W_i^1$ where $W_i^0 \cap W_i^1 = \emptyset$ and both W_i^0 and W_i^1 are infinite. Fix $i < \alpha$ for a moment. We know that

$$X = \left(X \cap \bigcup_{n \in W_i^0} g_i^{-1}(\{n\}) \right) \cup \left(X \cap \bigcup_{n \in W_i^1} g_i^{-1}(\{n\}) \right).$$

Since $X \in \mathcal{U}_0$ and $\mathcal{U}_0$ is an ultrafilter, we have that either $X \cap \bigcup_{n \in W_i^0} g_i^{-1}(\{n\}) \in \mathcal{U}_0$ or $X \cap \bigcup_{n \in W_i^1} g_i^{-1}(\{n\}) \in \mathcal{U}_0$. Suppose that $Y = X \cap \bigcup_{n \in W_i^0} g_i^{-1}(\{n\}) \in \mathcal{U}_0$ (the other case would be handled similarly). Note that by the definition of $\mathcal{U}_0$ we know that $a_0 \in {}^*Y$. Define $f_i : \omega \to \omega$ as follows: for $n \in Y$ let $f_i(n) = g_i(n)$, while for $n \notin Y$ let $f_i(n) = W_i^1(n)$. Now that functions f_i are defined, unfix i. We will prove that $\mathcal{F} = \{f_i : i < \alpha\}$ has all the properties from the conclusion of the lemma. Since each g_i ($i < \alpha$) is finite-to-one, it is clear that f_i is also finite-to-one. Again, this is because g_i is finite-to-one on ω, and outside of Y the function f_i is defined so that it is one-to-one. Since ${}^*g_i(a_0) = a_i$ and $a_0 \in {}^*Y$, it must be that ${}^*f_i(a_0) = a_i$ for each $i < \alpha$. Now we prove the last property. Suppose that $i < j < \alpha$. Since the set $X \setminus X_{ij}$ is finite and $Y \subseteq X$, there is $l < \omega$ so that $Y \setminus l \subseteq X_{ij}$. Take $m, n \geq l$, and suppose that $f_i(n) = f_i(m)$. There are three cases. First, $n, m \notin Y$. In this case, $f_i(n) = f_i(m)$ implies that $W_i^1(n) = W_i^1(m)$, i.e. $n = m$. Hence $f_j(n) = f_j(m)$. Second, $m \in Y$ and $n \notin Y$. Since $m \in Y$, $g_i(m) = f_i(m) = f_i(n)$ so $f_i(n) \in W_i^0$. On the other hand, since $n \notin Y$, $f_i(n) = W_i^1(n)$. Thus we have $f_i(n) \in W_i^0 \cap W_i^1$ which is in contradiction with the fact that $W_i^0 \cap W_i^1 = \emptyset$. So this case is not possible. Third, $m, n \in Y$. In this case $f_i(n) = f_i(m)$ implies that $g_i(n) = g_i(m)$. Since $m, n \in Y \setminus l \subseteq X_{ij}$ it must be that $f_j(n) = g_j(n) = \pi_{ij}(g_i(n)) = \pi_{ij}(g_i(m)) = g_j(m) = f_j(m)$ as required. Thus the lemma is proved. $\square$

Lemma 18 (MA_α). *Let $\langle M_i : i < \alpha \rangle$ be a $\subseteq$-decreasing sequence of principal, and pairwise cofinal submodels of M. Suppose that $\mathcal{U}_0 = \{X \subseteq \omega : a_0 \in {}^*X\}$ is a $P_\mathfrak{c}$-point, where a_0 generates M_0. Then there is an element $c \in \bigcap_{i<\alpha} M_i$ which generates a principal model cofinal with all M_i ($i < \alpha$).*

Proof. Let a_i for $i < \alpha$ be an element generating M_i. By Lemma 17 there is a family $\mathcal{F} = \{f_i : i < \alpha\} \subseteq \omega^\omega$ of finite-to-one functions such that ${}^*f_i(a_0) = a_i$ for $i < \alpha$, and that for $i, j < \alpha$ with $i < j$, there is $l < \omega$ such that $f_j(n) = f_j(m)$ whenever $f_i(n) = f_i(m)$ and $m, n \geq l$. By Lemma 7, for each finite $A \subseteq \alpha$, and each $n < \omega$, there is a finite $(A, \mathcal{F})$-closed set containing n as an element. Now using MA_α, Lemma 12 implies that there is a finite-to-one function $h \in \omega^\omega$, and a collection $\{e_i : i < \alpha\} \subseteq \omega^\omega$ such that for each $i < \alpha$ there is $l < \omega$ such that $h(n) = e_i(f_i(n))$ whenever $n \geq l$.

Let $c = {}^*h(a_0)$, and let M_α be a model generated by c. By Lemma in [1, pp. 104], M_α is cofinal with M_0. Thus M_α is a principal model cofinal with all M_i ($i < \alpha$). To finish the proof we still have to show that $c \in \bigcap_{i<\alpha} M_i$. Fix $i < \alpha$. Let $l < \omega$ be such that $h(n) = e_i(f_i(n))$ for $n \geq l$. If $a_0 < l$, then $M_j = M$ for each $j < \alpha$ so the conclusion is trivially satisfied. So $a_0 \,{}^*{\geq}\, l$. Since the sentence $(\forall n)[n \geq l \Rightarrow h(n) = e_i(f_i(n))]$ is true in M, it is also true in M_0. Thus $c = {}^*h(a_0) = {}^*e_i({}^*f_i(a_0)) = {}^*e_i(a_i) \in M_i$ as required. □

Theorem 19 (MA_α)**.** *Let M_i $(i < \alpha)$ be a collection of pairwise cofinal submodels of M. Suppose that M_0 is principal, and that $\mathcal{U}_0 = \{X \subseteq \omega : a_0 \in {}^*X\}$ is a $P_{\mathfrak{c}}$-point, where a_0 generates M_0. Then $\bigcap_{i<\alpha} M_i$ contains a principal submodel cofinal with each M_i.*

Proof. We define models M_i' for $i \leq \alpha$ as follows. $M_0' = M_0$. If M_i' is defined, then M_{i+1}' is a principal submodel of $M_i' \cap M_{i+1}$ cofinal with M_i' and M_{i+1}. This model exists by Corollary in [1, pp. 105]. If $i \leq \alpha$ is limit, then the model M_i' is a principal model cofinal with all M_j' $(j < i)$. This model exists by Lemma 18. Now the model M_α' is as required in the conclusion of the lemma. □

Theorem 20 (MA_α)**.** *Suppose that $\{\mathcal{U}_i : i < \alpha\}$ is a collection of P-points. Suppose moreover that $\mathcal{U}_0$ is a $P_{\mathfrak{c}}$-point such that $\mathcal{U}_i \leq_{RK} \mathcal{U}_0$ for each $i < \alpha$. Then there is a P-point $\mathcal{U}$ such that $\mathcal{U} \leq_{RK} \mathcal{U}_i$ for each i.*

Proof. By Theorem 3 of [2], $\omega^\omega/\mathcal{U}_i$ is isomorphic to an elementary submodel M_i of $\omega^\omega/\mathcal{U}_0$. Since all $\mathcal{U}_i$ $(i < \alpha)$ are non-principal, each model M_i $(i < \alpha)$ is non-standard. By Corollary in [2, pp. 150], each M_i $(i < \alpha)$ is cofinal with M_0. This implies that all the models M_i $(i < \alpha)$ are pairwise cofinal with each other. By Theorem 19 there is a principal model M' which is a subset of each M_i $(i < \alpha)$ and is cofinal with M_0. Since M' is principal, there is an element a generating M'. Let $\mathcal{U} = \{X \subseteq \omega : a \in {}^*X\}$. Then $\omega^\omega/\mathcal{U} \cong M'$. Since M' is cofinal with M_0, M' is not the standard model. Thus $\mathcal{U}$ is non-principal. Now $M' \prec M_i$ $(i < \alpha)$ implies that $\mathcal{U} \leq_{RK} \mathcal{U}_i$ (again using Theorem 3 of [2]). Since $\mathcal{U}$ is Rudin-Keisler below a P-point, $\mathcal{U}$ is also a P-point. □

Corollary 21 (MA)**.** *If a collection of fewer than $\mathfrak{c}$ many $P_{\mathfrak{c}}$-points has an upper bound which is a $P_{\mathfrak{c}}$-point, then it has a lower bound.*

Corollary 22 (MA)**.** *The class of $P_{\mathfrak{c}}$-points is downwards $< \mathfrak{c}$-closed under $\leq_{RK}$. In other words, if $\alpha < \mathfrak{c}$ and $\langle \mathcal{U}_i : i < \alpha \rangle$ is a sequence of $P_{\mathfrak{c}}$-points such that $\forall i < j < \alpha\, [\mathcal{U}_j \leq_{RK} \mathcal{U}_i]$, then there is a $P_{\mathfrak{c}}$-point $\mathcal{U}$ such that $\forall i < \alpha\, [\mathcal{U} \leq_{RK} \mathcal{U}_i]$.*

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